Valuation Of Harmonic Current Injections

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ABSTRACT

The number of power electronic and distorting loads, continue to grow at a rapid pace. Accompanying this growth in distortion sources is an increased sensitivity of certain types of loads to harmonic distortion, and poor power quality in general. The result is greater variance in the value loads place on a harmonic free supply, and therefore what they are prepared to pay to mitigate the potential consequences of harmonic distortion.

Harmonic distortion imposes costs worth considering, if not, no action would be taken by networks to mitigate the effects of harmonic injections. This thesis develops tools that allow the valuation of the harmonic injections made by loads throughout a network. The ability to accurately value harmonic distortion is critical if an optimal allocation of resources committed to the problem is to be achieved. Also this work develops methods by which an optimal allocation of resources can be brought about, and ways the costs of any action taken, can be distributed in a manner deemed fair.

Marginal pricing is the technique used to achieve an efficient allocation of resources. In a decentralised framework, marginal pricing will encourage efficient behaviour from each network participant. This is achieved by making the cost of each load's actions transparent, and borne by that load. Also marginal pricing fully utilises all the available knowledge throughout the system. The utilisation of knowledge is the key to solving all economic problems, and the difficulty associated with gathering knowledge makes centralised decision making inherently inefficient.

This thesis develops marginal prices for harmonic injections, and these prices are demonstrated to encourage efficient behaviour from each load with respect to reducing the injection they make into the system. It is also shown marginal pricing has the ability to encourage efficient allocation of filter resources. By determining exactly how much the distortion is worth, it is possible establish exactly how much the network is willing to pay, to reduce that distortion. There are multiple ways marginal pricing can be implemented, depending on whether charges are based on the Norton injections of each load or the total harmonic injection. The advantages and disadvantages of each method are discussed.

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GLOSSARY

General Notation

In the following X and W are generic variables.

X_i	Value of X at busbar i
$\mathbf{X}$	Vector of X values throughout the network
$\underline{\mathbf{X}}$	Vector of X values over space and time
X_{ij}	Element (i, j) in matrix
$[X]$	Matrix form of X
$[[X]]$	3D matrix form of X extending over different time periods
$\widetilde{X}$	X is a complex variable
$Re\{\widetilde{X}\}$	Real part of $\widetilde{X}$
$Im\{\widetilde{X}\}$	Real part of $\widetilde{X}$
$\widetilde{X}^*$	Conjugate of $\widetilde{X}$
$X, \|\widetilde{X}\|$	Magnitude of $\widetilde{X}$
$\angle\widetilde{X}$	Angle of $\widetilde{X}$
ΔX	Change in X
e^X	Exponential of X
$[e^{\mathbf{X}}]$	Diagonal matrix containing elements of $e^{\mathbf{X}}$
$\frac{\partial W}{\partial X}$	Partial derivative of W with respect to X
$W = f(X)$	W is a function of X

Symbols

α	Angle of harmonic current injected into the network by nonlinear loads
$\beta_{i,j}$	Phase angle of element i, j of harmonic admittance matrix
$\beta_{i,j}^{-1}$	Phase angle of element i, j of harmonic inverse admittance matrix
$\gamma_F(t)$	Marginal cost of generation fuel
$\gamma_M(t)$	Marginal cost of generation maintenance
$\gamma_{QS}(t)$	Marginal cost of generation quality of supply
$\gamma_R(t)$	Generation revenue reconciliation
$\eta_{L,k}(t)$	Network marginal losses
$\eta_{QS,k}(t)$	Network quality of supply
$\eta_{R,k}(t)$	Network revenue reconciliation
λ	Lagrange multiplier

θ	Angle of harmonic voltage
$\widetilde{\mu}_h$	Marginal price of harmonic injections, for basic network models
$\widetilde{\mu}_h^s$	Marginal price of harmonic injections, when passive filters are excluded
$\widetilde{\mu}_C$	Marginal price for passive filter capacity
$\widetilde{\mu}_N$	Marginal price of Norton injections
$\widetilde{\mu}_{Ti}$	Marginal price of harmonic injections
$\rho_k(t)$	Marginal price for energy at busbar k
ρ	Marginal cost of harmonic reduction (where reduction cost is linear)
ς	Angle of passive filter admittance
τ	Number of hours load injects harmonic current into the network
ψ	Angle of active filter current injected into network
$B[\mathbf{d}(t)]$	Customer benefit from consumption of energy
$\widetilde{C}$	Admittance of passive filter
C_{max}	Passive filter capacity
$d_i(t)$	Demand at busbar i
$d(t)$	Total demand
D_I	Current distortion power
D_V	Voltage distortion power
$E[x]$	Expected value of x
F	Marginal cost of active filter capacity
$FF(I_{fmax})$	Cost of capital for active filter
$FV(I_f)$	Operating cost of active filter
$g_i(t)$	Generation at busbar i
$g(t)$	Total demand
$G[\mathbf{g}(t)]$	Generation costs and constraints
h	Harmonic order
$[I]$	Identity matrix
I_{HA}	Harmonic adjusted RMS current
$\widetilde{I}_h$	Complex harmonic current initially injected into the network
$\widetilde{I}_N$	Complex Norton current injected into the network
$\widetilde{I}_R$	Reduction in complex harmonic current initially injected into the network
$\widetilde{I}_f$	Complex active filter current injected into the network
I_{fmax}	Active filter capacity
I_{RMS}	Total RMS current
$\widetilde{I}_T$	Total complex injections into the network
j	$\sqrt{(}-1)$
k_i	Load i's marginal valuation of voltage distortion if $u_i(V_{hi})$ is linear
k_i^*	Load i's stated marginal valuation of voltage distortion (linear utility function assumed)
$\mathbf{K}$	Marginal harmonic utility vector where utility function are linear
$\mathbf{K}^*$	Stated marginal harmonic utility vector (linear utility functions assumed)
M_i	Single point measurement quantity
n	Number of busbars in the network

N	Harmonic order
$N[\mathbf{z}(t)]$	Network costs and constraints
P	Real power
P_n	Weighting of frequency $50n$ in psophometric weighting table
$PF(C_{max})$	Cost of capital for passive filter
$PV(C)$	Operating cost of passive filter
Q	Reactive power
$RC(I_R)$	Harmonic reduction cost
S	Apparent power
S_1	Fundamental apparent power
S_H	Harmonic apparent power
S_N	Non-fundamental apparent power
t	Time
$u_i(V_{hi})$	Harmonic utility of load at busbar i
$U(\mathbf{V_h})$	Aggregate network harmonic utility
V_1	Fundamental RMS voltage
V_H	Harmonic RMS voltage
V_{HA}	Harmonic adjusted RMS voltage
$\widetilde{V}_h$	Complex harmonic voltage
V_{RMS}	Total RMS voltage
$x + jy$	Arbitrary complex number
$[Y_f(\widetilde{\mathbf{C}})]$	Filter admittance matrix for harmonic order h
Y_1, Y_2	Elements of admittance tensor in Norton models of nonlinear loads
$[Y_h]$	Admittance matrix for harmonic order h
$[Y_h]^{-1}$	Inverse admittance matrix for harmonic order h
$\widetilde{y}_{ij}$	Element (i, j) of harmonic admittance matrix
$\widetilde{y}_{ij}^{-1}$	Element (i, j) of harmonic inverse admittance matrix
$\mathbf{z}(t)$	Vector of energy line flows

Abbreviations

AC	Alternating Current
ACC	Annual Cost of Capital
dPF	Distortion Power Factor
DC	Direct Current
EDV	Equivalent Disturbing Voltage
HVDC	High Voltage Direct Current
IEEE	Institute of Electrical and Electronic Engineers
PF	Power Factor
pu	Per Unit
RMS	Root Mean Square
tPF	True Power Factor

THD	Total Harmonic Distortion
THD_I	Total Harmonic Current Distortion
THD_V	Total Harmonic Voltage Distortion
VAr	Volt Amp Reactive (Unit of reactive power)

Chapter 1

INTRODUCTION

1.1 GENERAL

Like the consumers of any good or commodity, the consumers of electrical energy are interested in the quality of their purchase. Not all power is of equal quality, and the key components of power quality are:

- **Voltage Stability** The voltage at each busbar is predictable in magnitude and not subject to periodic dips.
- **Frequency Stability** The frequency at each busbar is predictable and does not suffer from long or short term fluctuations.
- **Harmonic Distortion** The voltage waveform at each busbar is predictable and does not suffer excessive steady state or transient harmonic distortion.
- **Security of Supply** The power supply has a high level of reliability, so that loads can be assured their demands will be met as required.

The area of power quality considered in this thesis is harmonic distortion. Increased proliferation of power electronic devices throughout most electrical networks has seen an increase in the level of harmonic current injected into each network. This has increased the potential for harmonic distortion to negatively affect loads throughout the network, or the network itself. Compounding this problem is that the sensitivity of many loads to poor power quality has also increased. In particular the electronic automation of most industrial process and the computerisation of most workplaces, has increased the value some loads place on power quality. Some loads though (such as heating loads), are relatively insensitive to power quality and are likely to remain so, the result being an increased divergence of the power quality needs of the most sensitive loads and the most robust loads.

In dealing with harmonic distortion there are six basic steps which need to be undertaken:

1. Measurement of the harmonic distortion. Without the ability to quantify the level of distortion, any mitigation efforts are likely to prove inefficient and futile.
2. Assessment of whether the harmonic distortion present meets the requirements of the network. Specifically is there a problem? The cost impact of the harmonic injections needs to be assessed. This will include the impact on tranmission and distribution of equipment, along with the impact on customers.
3. Decision as to what level of distortion is desirable. If it is decided that the level of harmonic distortion present does not meet the needs of the network, a decision as to the level of distortion acceptable must be made.

4. Decision on the mitigation method to be employed. Having decided the need to reduce the level of distortion present, technical decisions as to what is the most effective way to achieve this reduction must be made.

5. Implementation of mitigation techniques. This will include the decision of who is responsible for taking the appropiate action.

6. Assessment of the harmonic management undertaken. The consequences of any action taken must be monitored and reviewed so that future situations benefit from past experiences.

1.2 THESIS OBJECTIVES

This thesis relates to items two, three and four in the above list. Specifically this work attempts to develop techniques that theoretically allow an efficient allocation of resources for mitigation of harmonic distortion. Also it works towards finding methods by which the costs associated with the harmonic mitigation can be distributed in a manner perceived as fair amongst the power system participants.

In achieving an efficient allocation of resources, there is a need to quantify the harmonic problem. A valuation must be placed on harmonic distortion. Much of the work in this thesis centres on how to value the harmonic distortion present. Then, given what the distortion is worth and the available means by which it can be mitigated, it must be determined what is the best course of action to take. Having found an efficient resource allocation the final question is how can this allocation be brought about.

Any effort to mitigate harmonic resources will cost money, and the cost must ultimately be borne by the loads in the network. The loads therefore ultimately have a collective interest in seeing that whatever action taken is efficient. Individually though, they have an interest in seeing that they bear costs which are no more than the benefits received. As such, a critical component of any resource allocation system is the perception by the network participants of 'fairness' in the allocation of harmonic mitigation costs.

In meeting the above requirements this thesis explores the use of marginal pricing.

1.3 THESIS OUTLINE

Chapter 2 contains a brief description of the costs associated with harmonic distortion and the techniques used to manage harmonic distortion. Some of the techniques developed to allocate harmonic costs are detailed. The move towards value based network planning is mentioned along with the marginal pricing of energy.

Chapter 3 details the development of the harmonic marginal prices for the simplest possible network model. The behaviour of the prices are characterised as the network parameters are varied. It is demonstrated, for this simple case, that the marginal prices provide the required incentives for an efficient allocation of resources. The different ways harmonic property rights can be allocated and the consequences of this for marginal pricing are covered. Moreover one of the weaknesses in marginal pricing of harmonic injections is examined. An example showing the improvements in network and individual welfare which can result from the implementation of marginal pricing is presented.

Chapter 4 looks at the resultant payments made by each loads, when marginal pricing is implemented. It considers how to deal with loads being charged a complex amount for their injections.

It also provides an interpretation of such payments. The variation in individual payments is detailed as the injections of other loads are varied.

Chapter 5 considers the inclusion of an active filter in the network. Characterisation of the optimal active filter resource allocation is developed and marginal prices are shown to potentially encourage such an allocation of resources. Circumstances that will prevent marginal pricing achieving an efficient allocation of harmonic resources are detailed. A comparison of the marginal pricing and Toll road pricing, where there is an active filter included in the network, is detailed.

Chapter 6 looks at marginal harmonic pricing where a passive filter can be included in the network. The conditions that describe the optimal allocation of passive filter resources are developed and it is shown marginal pricing has the potential to bring about such an allocation. The potential for filter owners to exploit any market power they posses, is investigated and the consequences of such market power exploitation are determined. An example of marginal pricing where passive filters are included in the network is presented.

Chapter 7 considers the consequences of modelling the nonlinear loads as voltage dependent current sources or as Norton equivalents (compared with only a fixed current source). The implications for marginal pricing, of loads' nonlinear injections being dependent on the distortion seen at the local busbar, are considered.

Chapter 8 contains the conclusions of this thesis looking at the problem from a 'knowledge utilisation' point of view. Some areas of future work which can improve and extend the material presented in this thesis are detailed.

Chapter 2

BACKGROUND

2.1 INTRODUCTION

Ideally the supply voltage should be undistorted at each busbar through out the network. A distortion free supply restricts the possible voltage waveform seen by all equipment to a sine wave of fundamental frequency, with a bounded amplitude. Such a highly specified supply voltage would ease the design process of all electrical equipment. It would also improve the operational performance of all equipment as the more predictable the supply voltage, the less likely it is to fall outside the design parameters of any piece of equipment. But this ideal state cannot be achieved as harmonic currents are injected into the system from a variety of sources. The main sources being large power converters (such as those used for HVDC links), large industrial sites, and increasingly large commercial premises. So that designers and users of equipment have some assurance as to the quality of voltage supply and hence performance of their products, harmonic standards or regulations are usually stipulated for any given network. These can vary from network to network and often take the form of maximum allowable harmonic current injections as a function of short circuit ratio ($\mathbf{I_{sc}/I_L}$), for different harmonic orders as shown in Table 2.1.

Table 2.1 IEEE 519 Harmonic Curent Limits

$\mathbf{I_{sc}/I_L}$	$\mathbf{h < 11}$	$\mathbf{11 \leq h < 17}$	$\mathbf{17 \leq h < 23}$	$\mathbf{23 \leq h < 35}$	$\mathbf{35 \leq h}$	**TDD**
$\mathbf{< 20}$	4.0	2.0	1.5	0.6	0.3	5.0
$\mathbf{20 - 50}$	7.0	3.5	2.5	1.0	0.8	8.0
$\mathbf{50 - 100}$	10.0	4.5	4.0	1.5	0.7	12.0
$\mathbf{100 - 1000}$	12.0	5.5	5.0	2.0	1.0	15.0
$\mathbf{> 1000}$	15.0	7.0	6.0	2.5	1.4	20.0

Also often stipulated is the maximum harmonic voltage content present at any busbar as a function of harmonic order. An example of such a regulation is shown in Table 2.2.

Finally there are composite measures for both the current injections and voltage harmonics such as Total Harmonic Distortion Voltage/Current, and Equivalent Disturbing Voltage/Current. These measures are intended to limit the combined effect of all the harmonics acting together. The expressions for the voltage measures are given in equations 2.1 and 2.2, the current equations are equivalent except the voltage terms are replaced by current terms.

$$THD = \frac{1}{V_1}\sqrt{\sum_{n=2}^{\infty} V_n^2} \tag{2.1}$$

Table 2.2 NZCEP 36 Harmonic Voltage Limits

Harmonic Order	Harmonic Voltage Level (As a percentage of nominal phase voltage)
3	2.3
5	1.4
7	1.0
9	0.8
11	0.7
13	0.6
15	0.5
17-21	0.4
23-49	0.3

$$EDV = 6.25 \times 10^{-5} \sqrt{\sum_{n=2}^{50} (nP_nV_n)^2} \tag{2.2}$$

Where P_n is the weighting given to the frequency 50n in the psophometric weighting table. One well known standard incorporating such rules is IEEE 519-1992 [519-19921993], though various equivalents are in place all over the world, examples being that which exist in Argentina [SanRoman and Ubeda1998], and used to exist in New Zealand [361993].

To this point these standards have provided a level of harmonic power quality which has satisfied most loads. Given this fact one may ask why not stay with the status quo, if adequate? The problem with the standards is that they give no consideration towards efficiency. They fail to consider if it is actually worth while for a given load to clean up their injections. Considering the growth in distorting loads such as power electronics, adjustable speed drives and computing equipment [McGranaghan and Mueller1999] [Makram *et al.*1993] [Emanuel *et al.*1995] dealing with the consequences of harmonic distortion is likely to become more important in the future. Considering the large expense associated with power system investment and the fact that someone must pay for what ever actions are taken to deal with any harmonic distortion, it stands to reason that tools need to be developed so that harmonics can be dealt with in the most efficient manner possible.

2.2 HARMONIC DISTORTION COSTS

The implicit assumption all this work is that harmonic distortion does have some detrimental effects on equipment and network participants. If not then the distortion seen by loads would be worthless and one would make no effort in trying to limit harmonic injections. But harmonics do have an affect, with different types of equipment affected to different degrees and with different results.

2.2.1 Capacitors

With capacitors the presence of harmonics results in additional heating and dielectric stress. Capacitors are also of interest in that they can have a substantial effect on the harmonic voltage through out the network. The capacitor can interact with the network so to cause a resonance at a frequency where harmonics are present. This resonance can result in harmonic voltage many times greater than may of otherwise been expected and is likely to result in blown fuses for the capacitor bank [Wagner *et al.*1993].

2.2.2 Circuit Breakers and Fuses

Severe harmonic distortion can influence the operation of circuit breakers and fuses. Any device which operates on the basis of a thermal mechanism, will be influenced by harmonic currents. It has been suggested that fuses and some circuit breakers may have there operating point shifted due to the extra heating caused by harmonic currents [Brozek1990]. Though it can be argued that these devices are RMS devices and that the harmonics do not shift the operating point at all, but instead change the RMS current from what is suggested by the fundamental current alone.

2.2.3 Electronic Equipment

Considering the role that electronics and power electronics have in almost all commercial and industrial processes, the effects harmonic distortion can have on electronic equipment have the most serious consequences. Should the voltage distortion be such that extra zero crossings exist any equipment that uses the zero crossing as a trigger is likely to misfire [Wagner *et al.*1993].

Harmonics will have an effect on the peak voltage seen by any electronic equipment. Should these peaks be larger than designed for, the resultant over voltages have the potential to damage some equipment. Harmonics are equally capable of flattening voltage waveforms. In power electronic circuits where the voltage peaks are used to charge a DC bus capacitor a flattening of the voltage waveform will mean the electronic equipment will have a DC under voltage condition. Therefore its operation will be more susceptible to any further under voltage conditions on the supply side.

Even without extra zero crossing a distorted harmonic voltage will have effects on the operation of equipment such as large power converters. The firing angle of thyristors will become modulated and may be altered compered to the non-distorted case. This will alter the characteristic harmonic injections of the converter and possibly result in the injection of non-characteristic harmonics. This alteration of switching instants will effect any power electronic equipment and may reduce the performance of equipment such as thyristor controlled reactors, if not accounted for [Montano *et al.*1993].

2.2.4 Metering

Many modern RMS meters are relatively unaffected by harmonic distortion. But induction disk meters (the most common type of meter in use) are likely to be effected by harmonic distortion. These have been found to often read high when the load being metered consists of harmonic generating equipment [Arrillaga *et al.*1985]

2.2.5 Transformers

Harmonics voltages increase iron losses and can cause increased stress on insulation. Where the voltage peak is increased, partial discharge between windings is possible. Harmonic currents are likely to increase copper losses [Arrillaga *et al.*1985], this increased heating can reduce the life expectancy of the transformer. Another consideration is that should a transformer with a delta wound secondary feed a highly distorted load, triplen harmonics will circulate in the delta winding, resulting in increased loading on this winding.

Transformers can also act a source of harmonic currents due to their non-linear magnetisation characteristics. The harmonic generation will increase substantially should it supply an asymmetrical load, as any DC load current will result in saturation of the magnetic circuit.

2.2.6 Rotating Machines

Harmonic distortion will cause temperature rise in rotating machines [Fuchs *et al.*1987] [Fuchs *et al.*1986], due to additional losses in the stator windings and rotor circuits. These temperature rises due to harmonic currents are likely to be localised causing hot spots, which will reduce the operating life of the machinery.

Harmonics have little affect on the mean torque produced by machines. But harmonic currents within a machine are likely to produce a pulsating air gap flux, which in turn will produce a pulsating torque. Consideration of these torque must be made to ensure that they do not excite any mechanical resonances.

Another potential problem exists in harmonic distortion may stop a machine coming up to speed due to ripples in the torque/speed characteristic.

2.2.7 Protective Relaying

Relaying can be affected by the presence of harmonic currents under fault conditions. In particular the operation of distance relays can be effected as measurement of the fundamental impedance can be subject to large errors in the presence of a highly distorted current [Arrillaga *et al.*1985]. This can result in costly false trippings.

Another effect is the false operation of some overcurrent relays, as relays with a simple rectifier input circuits respond to the rectified peak of the input rather than the RMS value.

2.2.8 Cumulative Costs For Utility and Load

Given all the above effects that harmonic voltages and currents have on the equipment owned by utilities and loads what is the total cost or value of harmonic distortion to loads and the utility serving them? Very little work has taken place in an attempt to answer this question. On the utility side it has been suggested *Pileggi et al* [Pileggi *et al.*1995] that the cost to utilities of harmonic distortion may be very little. In this study the author attempted to find the cost (looking forward from 1990 to 2010) that harmonic injections have in terms of additional energy losses and capital costs associated with mitigation equipment installed, to a utility operating in the Northeast United States. Mitigation equipment was installed on the basis that THD_V does not exceed 5% at any busbar (broadly in line with IEEE-519). A number of different feeder configurations and load forecasts were used, as depending on the growth of different type of harmonic load and the feeder configuration the resultant harmonic voltage could vary considerably. The cost of distortion was found to be as low as US$2.00/KVA/year in cases where no mitigation equipment was required to be installed. But in cases where mitigation equipment was required, or where there was a short supply of generating capacity (so that the harmonic losses became valuable) the cost of harmonics were liable to increase by two orders of magnitude. It is worth noting that this study found that majority of the costs resulted from installing mitigation equipment to satisfy harmonic standards. Hence for an efficient allocation of resources with respect to harmonics, the basis for deciding on the level of mitigation equipment required is critical. The basis on which this is presently done is to comply with the regulations or standard in place. But working to any standard cannot produce an efficient allocation of resources where there are large disparities in how different loads value voltage distortion.

Unfortunately there seems to have been little work done with respect to determining the value of harmonic distortion to different types of loads. However there have been some studies of the cost interruption of supply has to different types of load [Sullivan *et al.*1996] [Sullivan *et al.*1997]. It is thought by this author, that interruption of supply can act as a reasonable proxy for harmonic distortion at least when trying to compare relative costs between different load groups.

Moreover using interrupted supply as a proxy can be justified in the sense that severe distortion can have the effect of making equipment temporarily inoperable, which has the same consequences as a loss of supply. These studies found a very large disparity between the valuation residential, commercial and industrial loads put of a loss of supply. Not surprising the valuation individual residential loads, place on any loss of supply is small (in the vicinity of US$5.00/event). While commercial loads had valuations that are two orders of magnitude above those of residential loads, and industrial loads were another order of magnitude larger again. It was noted that there is considerable variation within the industrial customers as to how they valued an interruption ranging from close to nil to more than US$1million per event. From these studies one can see the value different loads put on their electrical supply has a variation of six orders of magnitude. From this it should be safe to conclude that there will be a similar variations on how harmonic distortion is valued.

Given that 'what is good for the goose may not be good for the gander', the question can asked, if harmonics are to be controlled by regulations or standards how are these to be determined. It seems to be the case these standards are such that most loads will not suffer any ill effects from harmonic distortion. But this cannot be efficient in the face of such variable valuations of distortion, as those few loads which are sensitive to the distortion allowed, may be those with the highest resultant costs. Also it can be argued why should loads that are ambivalent to harmonic distortion be forced to pay for mitigation equipment to meet a standard far in excess of their requirements?

2.3 ALLOCATION OF HARMONIC COSTS

There have been a number of methods put forward for dealing with the costs generated by harmonic current injections. These range from the purely theoretical methods that assume complete knowledge of the system and the actions of each load in the system, to methods which can be implemented with the instrumentation currently available and installed in most systems.

2.3.1 Allocation of Harmonic Costs: The Complications

There are a number of characteristics of harmonic distortion that complicate the allocation of resultant costs [Emanuel1999]. Some of these characteristics include:

- Considerable range of harmonic orders which can effect the network.
- Loads may not be consistent injectors or absorbers of harmonics. A load which injects harmonic current of order i, may absorb harmonics of order j.
- Voltage distortion will effect the injections produced by non-linear loads and the injections in turn will have an effect on any voltage distortion present.
- Negative sequence (as a result of unbalanced loads) results in the generation of non-characteristic harmonics
- Both harmonic currents and voltages have consequences
- Different loads have different mechanisms that make them susceptible to distortion
- Existing equipment has been designed to existing standards and will still have a substantial remaining life

There can also be problems establishing who is responsible for the injections seen at the point of common coupling between the load and the network [Xu and Liu2000]. Figure 2.1 shows Xu's model of both the harmonic load and the utility it is connected to, where both the utility and load are represented as Norton equivalents. Given this representation should the Norton impedance looking into the network from the point of common coupling (Zu) change, then this will effect the harmonic current injected in to the network at the point of common coupling ($\widetilde{I}_{PCC}$). It can be argued that a load should not be responsible for changes on the utility side.

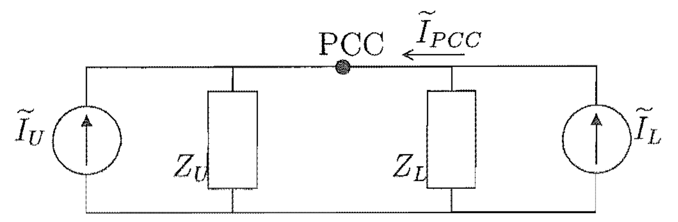


Figure 2.1 Harmonic equivalent circuit of a distorting load and network

There have been suggestions that charges for harmonic injections can be based on the flow of harmonic power. The assumption underlying this being that should a load be injecting harmonic energy into the network as shown in Figure 2.2, then that loads actions are aggravating the harmonic problem throughout the network. There are a number of problems with using the flow of real harmonic power as the basis for charges. The first is that the real harmonic power flows often tend to be very small. At harmonic frequencies the networks look overwhelmingly inductive, hence the voltages generated and current injections will be very close to quadrature. The vast majority of harmonic flows are reactive flows and to base charges on the real flows is a failure to identify what is causing the problem. Another reason why charges based on

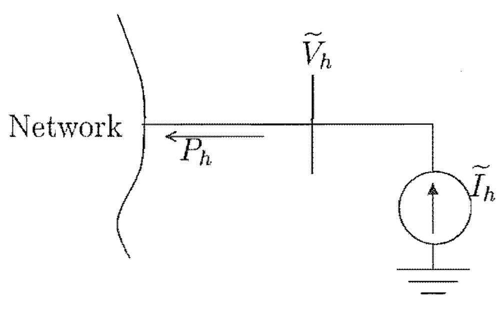


Figure 2.2 Harmonic power injection into a network

real current injections is not suitable is that is it possible to inject real harmonic power into the network ($Re\{\widetilde{V}_h\widetilde{I}_h^*\} > 0$) but not necessarily have any effect on the harmonic voltages throughout the network [Emanuel1995]. Should the network look inductive then any real harmonic power injection is likely to have little effect on the voltage throughout the network. Also the presence of resonances may mean injections, while lowering voltage at one busbar, may be aggravating the problem at another.

2.3.2 Allocation of Harmonic Costs: The Methods

It would be nice to have harmonic charges based on some single composite measure so that they are easily understood by all. Unfortunately harmonics are not well described by single composite numbers and many traditional measures such as reactive power and power factor, which were developed for fundamental frequencies become a bit impractical in the presence of harmonics. None the less some such allocations methods have been put forward.

Emanual [Emanuel1995] suggested a potential method to account for harmonic distortion based on separating the apparent power into the fundamental and the non-fundamental component.

$$S^2 = S_1^2 + S_N^2 \tag{2.3}$$

Where $S_1 = V_1 I_1 =$ the conventional fundamental apparent power, and the non fundamental apparent power is:

$$S_N = \sqrt{D_I^2 + D_V^2 + S_H^2} \tag{2.4}$$

$$\begin{array}{ccccl} D_I & = & V_1 I_H & = & \text{Current Distortion Power} \\ D_V & = & V_H I_1 & = & \text{Voltage Distortion Power} \\ S_H & = & V_H I_H & = & \text{Harmonic Apparent Power} \\ & & V_H^2 & = & \sum_{h\neq 1} V_h^2 \\ & & I_H^2 & = & \sum_{h\neq 1} I_h^2 \end{array}$$

By taking the ratio of equation 2.4 and S_1 the resultant is

$$\frac{S_N}{S_1} = \sqrt{(THD_I)^2 + (THD_V)^2 + (THD_V \cdot THD_I)^2} \tag{2.5}$$

Given that in many cases $THD_I \gg THD_V$ the expression in 2.5 can be generally be simplified to

$$S_N = (THD_I) S_1 \tag{2.6}$$

This suggests that if one were interested in using S_N, as the basis for harmonic charges, this amount is simply calculated. A problem lies in that THD_I gives no indication as to what effect the current is having on voltages through out the network. An active filter is likely to have a very high THD_I, but clearly they should not be penalised.

There was also work performed by McEarchern *et al* looking at different ways to allocate the cost of harmonics, each with their own characteristics and problems [McEachern *et al.*1995].

- **Regulation** - With regulations economic efficiency is unlikely, and there are issues as to the appropriate actions when a load fails to comply.

- **Rebate restrictions** - Rebate restrictions refers to the fact on some networks, rebates are offered to those who go to the expense of installing energy efficient loads. But these energy efficient loads are often distorting loads such as fluorescent lights or asynchronous speed drives. Hence one must make sure any rebate reflects the harmonic costs the energy efficient device imposes on the network.

- **Charge for KVA hours** - Charging for KVA hours is nice in that it allows one to wrap conventional power factor charges and harmonic charges together into one payment. The down side is that KVA hour charges fail to reflect the fact that higher order harmonics tend to be more costly and damaging than lower order.

- **Charge for D** - Where $S^2 = P^2 + Q^2 + D^2$ charge for D. This is the same as charges based on S_N as above, with the same problems. It is some times suggested the problem with charges based on D, is that D has little physical meaning. While technically true, as demonstrated by Emanuel, D generally corresponds pretty closely to THD_I.

- **Charge for true power factor** - Power factor can be calculated in two different ways, which are both equivalent if dealing with sinusoidal voltages and currents.

$$\text{Displacement Power Factor } dPF = \cos(\theta_V - \theta_I) \tag{2.7}$$

$$\text{True Power Factor } tPF = \frac{P}{V_{RMS} I_{RMS}} \tag{2.8}$$

  Often power factor charges are based on the dPF, and amount which has questionable meaning in the presence of harmonic distortion. The tPF will reflect harmonic distortion, but it has a weakness in that it treats all harmonic orders equally.

- **Harmonic adjusted watt hours**- Charge for power flow at each harmonic order eg. \$0.10c/kwh for fundamental and \$1.50c/kwh for fifth harmonic. Where the harmonic charges would be independent of power flow direction (for reasons explained previously). A problem exists in that non-distorting loads will still have harmonic flows, but clearly should not be charged. In fact passive resistive loads are likely to be those with the largest real harmonic power flows, because as mentioned before an active source injecting into an inductive network is likely to generate a voltage which is quadrature to the injections.

- **Harmonic adjusted power factor** - This was the preferred solution from MacEachern, it is similar to using the True Power Factor as above except that a weighted RMS voltage and current are used.

$$\text{Harmonic Adjusted Power Factor } hPF = \frac{P}{V_{HA} I_{HA}} \tag{2.9}$$

$$\text{Where } V_{HA} = \left[\sum C_N V_N^2\right]^{\frac{1}{2}} \tag{2.10}$$

$$I_{HA} = \left[\sum K_N I_N^2\right]^{\frac{1}{2}} \tag{2.11}$$

  N : Harmonic order
  C_N : Weighting term for voltage term of order N
  I_N : Weighting term for current term of order N

  The question then arrises what weighting terms should be used for the different harmonic orders. For current harmonics, suggestions have included $K_N = N, K_N = N^{1.333}, K_N = \sqrt{N}, K_N = \left[1 + x \cdot (N^2 - 1)\right]$. These weighting terms are chosen on the basis of convenience, are related to IEEE 519, or reflect the effect of reduced skin depth with increasing frequency.

Work has been performed to investigate the different harmonic measures as proposed by MacEachern [Arseneau1999]. Using sample data from three medium industrial sites the harmonic measures of Displacement Power Factor, True Power Factor and Harmonic Adjusted Power Factor ($K_N = N^{1.333}$ & $K_N = \sqrt{N}$) were calculated, the results of these calculations are shown in Table 2.3.

The first point to note is, despite the fact that both the current and voltages at the sites examined were distorted, the displacement power factor still measures unity. Considerable differences between the harmonic measures can be seen, therefore charges based on these measures

Table 2.3 Results of Different Single Point Harmonic Measures

	Harmonic Measure			
	dPF	tPF	hPF ($K_N = N^{1.333}$)	hPF ($K_N = \sqrt{N}$)
Site 1	1.00	0.97	0.92	0.42
Site 2	1.00	0.80	0.48	0.11
Site 3	1.00	0.77	0.44	0.11

will need to vary considerably based on which is chosen. One also finds the Harmonic Adjusted Power Factor is heavily dependent on the weighting chosen. This huge variability leads to the question of which measure is appropriate, as if one is appropriate the others certainly are not. The answer to this lies in what is the value of harmonic injections made by each load. A question that all these measures fail to answer.

Work on allocation of harmonic costs has also been undertaken by Davis *et al* [Davis *et al.*2000b]. Known as the Toll Road method it looks to obtain a fair allocation of harmonic costs based on the assumption costs associated with harmonics are proportional to the square of injected currents. In the Toll Road methods injections are separated into those that increase (I_{ph}) and those that decrease (I_{nh}) the harmonic current through a certain component of the network. So that the current through the component is given by

$$I_{ch} = I_{ph} - I_{nh} \tag{2.12}$$

Given the assumption that the costs associated with harmonic current flow are proportional to the square of the current, then the harmonic costs are:

$$\text{Harmonic Costs} \propto I_{ph}^2 + I_{nh}^2 - 2I_{ph}I_{nh} \tag{2.13}$$

The Toll Road method sparates the interaction contribution of the two positive and negative currents ($-2I_{ph}I_{nh}$), based on the square of the currents, so that equation 2.13 can be simplified to:

$$I_{ph}^2 + I_{nh}^2 - 2I_{ph}I_{nh} = A_h I_{ph}^2 + B_h I_{nh}^2 \tag{2.14}$$

Using this result the Toll Road method suggests that individual injectors should contribute to the costs of harmonic mitigation based on the proportion of the load they place on the equipment. In the case of a harmonic source which makes a positive contribution to the current through the component

$$\text{Harmonic Charges for Load } i = (ACC)\,\frac{\tau}{8760}\,\frac{\int_0^{\tau}\sum_{h\neq 1}\frac{A_h I_{pih}^2}{h}dt}{\int_0^{\tau}\sum_{h\neq 1}\frac{I_{ch}^2}{h}dt} \tag{2.15}$$

Where ACC : Annual cost of capital of equipment
τ : Number of hours load i injected harmonic current into the component

To allocate the costs for harmonic mitigation equipment according to the Toll Road method requires synchronous measurements of harmonic currents and voltage through out the network. Such measurements are not readily available throughout most networks, so Davis *et al* [Davis *et al.*2000a] looked at a number of single point measurements to see how close they could come to the idealised Toll Road Method in allocating costs. The single point measures considered were

- **Harmonic Active Power**

$$P_h = 3\sum_{h\neq 1} V_h I_h \cos(\theta_h) \tag{2.16}$$

- **Harmonic Apparent Power** - This is the same measure as described in equation 2.4. Where the three phase equivalent the measure is

$$S_H = 3V_H I_H \tag{2.17}$$

- **Non Fundamental Apparent Power** - Again as described in 2.4 with

$$S_N = \sqrt{S^2 - S_1^2} \tag{2.18}$$

- **Total Harmonic Current Squared** - I_H^2

- **Non Fundamental Apparent Power Squared** - S_N^2

Using these single point measurements the harmonic costs are allocated according to

$$\text{Harmonic Charges for Load } i = (ACC)\frac{\tau}{8760}\frac{M_i}{M_T} \qquad \text{Where } M_T = \sum M_i \tag{2.19}$$

$$M_i = \text{Single point measurement quantity}$$

It was found that using harmonic active power as the measurement quantity produces a cost allocation with a very low correlation with the Toll Road method. On the other hand using S_N^2, produces a cost allocation very similar to the Toll Road method. I_H^2 and S_N are also found to result in an allocation reasonably close to the Toll Road method. This suggests that if one is to accept the Toll Road method as the proper way to allocate costs associated with harmonic mitigation equipment (as is put forward by Davis *et al*), then it is possible to get results very close to those obtained from the Toll Road method without using synchronous measurements.

With the Toll Road method both loads whose injections are in phase with the prevailing voltages and those which are out of phase, make a contribution to any costs associated with harmonic mitigation. This runs contrary to economic conditions for efficient allocation of resources, which require that each individual face the costs of their actions. A paper by Bergeron [Bergeron and Slimani1999] goes part of the way to bridging the gap between the Toll Road Method and marginal pricing. It follows the same basic methodology as the Toll road method, however only loads whose injections are increasing the prevailing harmonic voltages are required to contribute to the costs associated with the harmonics. These costs are shared out amongst those loads, which make a positive contribution to the prevailing voltages on either a linear or quadratic basis. So that if

$$\begin{aligned} |I_i^+| &= \text{Projection of load } i \text{ current on to resultant} \\ T &= \text{Total costs to be paid} \end{aligned}$$

$$\text{Amount paid by load } i = \frac{|I_i^+|}{\sum_i |I_i^+|}T \quad \text{or} \quad \frac{|I_i^+|^2}{\sum_i |I_i^+|^2}T \tag{2.20}$$

2.4 VALUE BASED NETWORK PLANNING

As can be seen there have been a number of methods developed to produce a "fair" allocation of the costs associated with harmonic distortion. These costs being those associated with meeting a standard or regulation. But the allocation of costs is only the second half the problem, with the first half being determining what is the optimal network investment and action to be taken given present conditions. In other words all the previous methods lacked the ability to discover what is the efficient course of action to be taken with respect to any harmonic distortion which may exist.

The planning and investment process for transmission and distribution assets (not associated with harmonic mitigation), also used to be based on idea that the transmission and distribution networks should have sufficient capacity to withstand a set of pre-defined contingencies [Vojdani *et al.*1996]. The results being that systems were often over built to avoid disruption for low probability events or to avoid interruption of low value loads. But there has been a movement towards value based transmission planning in which there is an attempt to overcome the short comings associated with deterministic planning methods. In one example the value based approach [Dalton *et al.*1996], the following factors are taken into account.

- Likelihood of different contingencies (and likely duration)
- Likelihood of an over load given a contingency
- Value of lost load to different customers

Considering that all costs borne by the utility are ultimately borne by the customer, the idea is to minimise the customers' total costs, which consist of the utility costs and the customer outage costs. There are difficulties in attempting to under take such an investment methodology, these include establishing realistic estimates for network component outages along with estimates of customers valuation of lost load under different circumstances. But despite these difficulties this approach is one that attempts to find an efficient solution to the resource allocation problem. It suggests ideas, which the harmonic allocation methods should incorporate so that they are not only fair, but also efficient.

2.5 MARGINAL PRICING

The ideal system is one, which will both allocate resources efficiently and then distributes costs in a manner, which is perceived to be fair. Characteristics exhibited by the marginal pricing approach taken in economics. It can be shown that in the solution of a constrained optimisation problem yields a set of shadow prices which can be used to achieve an optimal allocation of resources, and distribute any costs/benefits fairly [Varian1992] [Intriligator1971]. With marginal pricing each individual is faced with the true marginal costs of their actions, and they are then free to make decisions based upon their own costs and benefits of each action. Marginal pricing has already been well developed with respect to electrical energy (real power), with much of the pioneering in this area was done by Schweppe *et al* [Schweppe *et al.*1988]. Schweppe's marginal energy prices can be expressed on a component basis as shown in equation 2.21

$$\rho_k(t) = \gamma_F(t) + \gamma_M(t) + \gamma_{QS}(t) + \gamma_R(t) + \eta_{L,k}(t) + \eta_{QS,k}(t) + \eta_{R,k}(t) \tag{2.21}$$

Where the components which make up the price are:

$$\begin{aligned}
\gamma_F(t):&\quad \text{Generation Marginal Fuel}\\
\gamma_M(t):&\quad \text{Generation Marginal Maintenance}\\
\gamma_{QS}(t):&\quad \text{Generation Quality of Supply}\\
\gamma_R(t):&\quad \text{Generation Revenue Reconciliation}\\
\eta_{L,k}(t):&\quad \text{Network Marginal Losses}\\
\eta_{QS,k}(t):&\quad \text{Network Quality of Supply}\\
\eta_{R,k}(t):&\quad \text{Network Revenue Reconciliation}
\end{aligned}$$

Schweppe's formulation of marginal energy prices was derived from the following Lagrangian:

$$\mathcal{L}(t) = G\left[\mathbf{g}(t)\right] + N\left[\mathbf{z}(t)\right] - B\left[\mathbf{d}(t)\right] + \mu_e(t)\left[d(t) + L\left[\mathbf{z}(t)\right] - g(t)\right] \tag{2.22}$$

Where
$$\begin{aligned}
G\left[\mathbf{g}(t)\right]:&\quad \text{Generation Costs and Constraints}\\
N\left[\mathbf{z}(t)\right]:&\quad \text{Network Costs and Constraints}\\
B\left[\mathbf{d}(t)\right]:&\quad \text{Customer Benefit}\\
\mu_e(t)\left[d(t) + L\left[\mathbf{z}(t)\right] - g(t)\right]:&\quad \text{Energy Balance Constraint}
\end{aligned}$$

The first order conditions of equation 2.22 yeilds equation 2.23.

$$-\frac{\partial B\left[\mathbf{d}(t)\right]}{\partial d_k(t)} + \frac{\partial N\left[\mathbf{z}(t)\right]}{\partial d_k(t)} + \mu_e(t)\left[1 + \frac{\partial L\left[\mathbf{z}(t)\right]}{\partial d_k(t)}\right] = 0 \tag{2.23}$$

Assuming that the customer acts rationally in that the marginal utility gained equals the marginal price paid for the energy equation 2.23, can be rewritten to show the market clearing marginal price for energy.

$$\rho_k(t) = \mu_e(t)\left[1 + \frac{\partial L\left[\mathbf{z}(t)\right]}{\partial d_k(t)}\right] + \frac{\partial N\left[\mathbf{z}(t)\right]}{\partial d_k(t)} \tag{2.24}$$

Calculating these prices is not a trivial task for any network of realistic size. To ease the calculation process Schweppe proposes the use of a DC load-flow, which looks at the real energy flows only. This cut down load-flow works reasonably well for large transmission systems. Though it is possible given modern computing technology to use a full AC load flow solution, which has the advantage of allowing reactive power pricing to be incorporated.

The marginal pricing should lead to an optimal allocation of resources in the presence of a perfect (or near to perfect) market. The existence of an efficient market is dependent on there being a limit of the economies of scale available in relation to the size of the market. Should the minimum efficient scale be of a similar size as the market, a natural monopoly is the result. The usefulness of marginal pricing techniques to achieve efficient outcomes for energy is hence dependent on the minimum efficient scale compared with the market size. Evidence suggests that a competitive market can operate in the energy industry, despite the very large economies of scale present, due to a huge market, and growth in that market which outpaces the growth in the minimum efficient scale [Green2000]. In fact with the development of combined cycle technology (compared with traditional single cycle coal plants) the minimum efficient scale has been shrinking.

Marginal pricing of energy is being increasingly implemented through out the world due to its afore mentioned desirable characteristics. This had lead to considerable research into the potential uses of marginal pricing into other parts of the power system. Work has been performed that suggests marginal pricing should be used for reactive power [Baughman and Siddiqi1991] to achieve efficient investment outcomes for utilities and their customers. Chattopadhyay *et* *al* [Chattopadhyay *et al.*1995] uses marginal pricing of reactive power as the basis for the recovery of the costs associated with providing reactive power, and there has been work performed on different ways reactive power pricing may be implemented in practice [Gil *et al.*2000].

The study of marginal pricing techniques has moved beyond real and reactive power and its use has been proposed for system security [Berger and Schweppe1989] [Kaye *et al.*1995] (although at this stage the short time frame over which prices have to be calculated can prove prohibitive with respect to using prices to coordinate system security). Finally there have been attempts to construct a 'complete' marginal pricing systems for electrical systems which incorporate real and reactive power along with system security [M.L.Baughman *et al.*1997] [Baughman *et al.*1997].

This work to date has developed a sound frame work on which the operation of many power systems are based. Specifically marginal pricing of real energy is used as the basis of generation dispatch in areas such as New Zealand, Australia and parts of the United States. Marginal pricing of reactive power or other power system quantities has yet be implimented in any system, and due to some of the associated time scale difficulties is unlikely to implimented in the near future. The rapid pace at which the marginal pricing of energy has been adopted serves as evidence the techniques have merit, and are worthy of implimentation in other areas.

2.6 CONCLUSION

The presence of harmonic currents and voltage through out an electrical network will have consequences for a range of equipment. Given the effects that harmonic distortion will have on the utility and different loads, which are almost always of a negative nature, actions have been taken to limit harmonic injections. Currently harmonics tend to be controlled by a set of regulations and standards. Unfortunately with increasing variance between the power quality requirements of different loads any single standard is increasing unable to deliver an efficient solution to the harmonic problem.

To this point there has been little interest in methods to solve for efficient harmonic solutions with much of the study going towards establishing 'fair' distributions of the cost generated by meeting whatever standard is in place. Unfortunately without a good measure of the value of harmonic injections, any attempt to allocate the costs of harmonic injections will always be somewhat arbitrary. It is also the case that unless the level of costs associated with harmonic injections is efficient no allocation of these costs can ever be fair. For these reasons having an arbitrary standard and then trying to fairly allocate the costs of meeting the standard is fundamentally flawed.

In both the areas of network planning and in particular energy delivery there have been movements towards value based allocation of resources. With respect to energy it has been shown the use of marginal pricing is effective in ensuring an efficient allocation of resources as each participant is faced with the true value of their actions. A fair allocation of costs is also a result as each participant is free to make their own decision based on knowledge of the costs and benefits of their own actions. Given these favourable properties this thesis explores the use of marginal pricing as a tool to properly value harmonic injections, and as method to optimise the harmonic resource allocation, along with fairly allocating the costs.

Chapter 3

BASIC PRICE DEVELOPMENT

3.1 INTRODUCTION

Harmonic distortion has some value to individual network participants and the existing methods of allocating the costs are unlikely to achieve an optimal allocation of resources. This chapter looks at the development of basic marginal harmonic prices which will both achieve an optimal allocation of resources along with allocating the costs in a manner deemed fair. The use of static optimisation techniques to develop energy prices is well established and has proved successful in making the value of individuals' actions transparent, allowing for an optimal allocation of resources. It is thought the use of marginal prices will have the same effect with harmonics. Some of the assumptions underlying the development of the prices are also outlined followed by the basic form of the prices. The properties of the basic prices are discussed, and it is shown that the prices do in fact provide incentives for the loads to optimise their behaviour. The use of harmonic prices can be shown to lead to an optimal allocation of resources, but it provides no information as to what the appropriate harmonic property rights might be. Harmonic pricing is shown to be of value no matter how those property rights are allocated. The developed marginal prices are a function of the different characteristics of each load in the network. How the prices behave as the characteristics of the loads within the network change is investigated. Finally one of the potential pitfalls associated with the use of marginal pricing is detailed.

3.2 UTILITY AND OPTIMISATION

The underlying motivation for this work is that harmonic distortion at a busbar has some sort of negative impact on the local load (discussed in Chapter 2). This isn't always true, and for this reason it is likely that marginal pricing need only be implemented in specific areas of a distribution system such as commercial or industrial centres where harmonic disturbances do have a value/cost (in this thesis there is assumed to be only one load at each busbar) and it is this cost the pricing system will try to minimise.

In order that the model is not overly complex this chapter considers only one harmonic order, however the model is easily extended to include others. Moreover with the appropriate models of the nonlinear loads one can also consider cross modulation between harmonics. It is assumed that the cost of harmonic distortion to a load is a function of the harmonic voltage magnitude at that busbar. This cost is expressed in the form of a utility function, which quantifies the negative benefit associated with harmonic voltages.

$$u_i(V_{hi}) = \text{Utility of the load at busbar i} \leq 0 \tag{3.1}$$

$$U(\mathbf{V_h}) = \sum_{i=1}^{n} u_{hi}(V_{hi}) = \text{Total system utility} \tag{3.2}$$

In optimising this utility function there is only one constraint, the system nodal equation:

$$\widetilde{\mathbf{I}}_\mathbf{h} = [Y_h]\widetilde{\mathbf{V}}_\mathbf{h} \tag{3.3}$$

Some assumptions have been made with respect to the variables that form this constraint. Initially will assume that the injections made by any nonlinear load are fixed and not subject to change (relaxed in Section 3.4). Also these injections are also assumed to be independent of the harmonic voltage at the busbar. It is also considered that the linear and nonlinear loads at each busbar are exogenous. In the short term no load is going to manage their operations around what they pay for harmonics. The consequences of these assumptions is that both $\widetilde{\mathbf{I}}_\mathbf{h}$ and $[Y_h]$ are fixed.

The Lagrangian for this system is:

$$\mathcal{L}(\mathbf{V_h}, \widetilde{\mu}_h) = U(\mathbf{V_h}) + \widetilde{\mu}_h(\widetilde{\mathbf{I}}_\mathbf{h} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}) \tag{3.4}$$

The first order conditions for this system are given by:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} - \widetilde{\mu}_h[Y_h]\frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} = \mathbf{0} \tag{3.5}$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\mu}_h} = \widetilde{\mathbf{I}}_\mathbf{h} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h} = \mathbf{0} \tag{3.6}$$

Given that:

$$\widetilde{\mathbf{V}}_\mathbf{h} = \begin{pmatrix} V_{h1}e^{j\theta_1} \\ V_{h2}e^{j\theta_2} \\ \vdots \\ V_{hn}e^{j\theta_n} \end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} = \begin{pmatrix} e^{j\theta_1} & 0 & \dots & 0 \\ 0 & e^{j\theta_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{j\theta_n} \end{pmatrix} = [e^{j\theta}] \tag{3.7}$$

It follows from equation 3.5 that the optimal prices for harmonic current injections are given by:

$$\widetilde{\mu}_h = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}[e^{j\theta}]^{-1}[Y_h]^{-1} \tag{3.8}$$

Should each load have a constant marginal utility resulting from harmonic distortion such that

$$\begin{aligned} U(\mathbf{V_h}) &= \textstyle\sum_{i=1}^{n} k_i V_{hi} \\ U(\mathbf{V_h}) &= \mathbf{K}\mathbf{V_h} \\ \text{Where } \mathbf{K} &= (k_1, k_2, \dots, k_n) \end{aligned} \tag{3.9}$$

Then equation 3.8 can be simplified to:

$$\widetilde{\mu}_h = \mathbf{K}[e^{j\boldsymbol{\theta}}]^{-1}[Y_h]^{-1} \tag{3.10}$$

3.3 CHARACTERISTICS OF $\widetilde{\mu}_H$

An important requirement of any prices developed is that the full cost of harmonic distortion is collected from those injecting harmonic currents. For the prices developed in equation 3.10 this is easily shown to be the case.

$$\begin{aligned}
\widetilde{\mu}_h \widetilde{\mathbf{I}}_\mathbf{h} &= \mathbf{K}[e^{j\boldsymbol{\theta}}]^{-1}[Y_h]^{-1}\widetilde{\mathbf{I}}_\mathbf{h} \\
&= \mathbf{K}[e^{j\boldsymbol{\theta}}]^{-1}[Y_h]^{-1}[Y_h]\widetilde{\mathbf{V}}_\mathbf{h} \\
&= \mathbf{K}[e^{j\boldsymbol{\theta}}]^{-1}[e^{j\boldsymbol{\theta}}]\mathbf{V}_\mathbf{h} \\
&= \mathbf{K}\mathbf{V}_\mathbf{h} \\
\text{Total payments} &= \text{Harmonic distortion costs}
\end{aligned} \tag{3.11}$$

This result is of interest as while in general the prices charged for harmonic injections will be complex, 3.11 shows the total amount collected from the distorting loads is a real number, and is equal to the total costs the distortions impose on other loads. There is an implicit assumption here that each load has a right to a pollution free supply. This need not be the case, the harmonic property rights can just as easily be assigned so that each load has a right to inject what it likes into the system, this is expanded upon in Section 3.5.

In general the prices given in 3.10 will be a complex number, which will vary depending on the harmonic utility functions of the loads, and the resulting harmonic voltage profile. In the case of constant marginal utility the only endogenous variable is the angle of the voltage harmonic at each busbar. As such knowledge on how the harmonic voltage angles behave, will provide information on how the prices will behave. The system nodal equation can be expressed as

$$[e^{j\boldsymbol{\alpha}}]\mathbf{I}_\mathbf{h} = [Y_h][e^{j\boldsymbol{\theta}}]\mathbf{V}_\mathbf{h} \tag{3.12}$$

Which implies

$$\begin{aligned}
\widetilde{\mathbf{V}}_\mathbf{h} &= [Y_h]^{-1}[e^{j\boldsymbol{\alpha}}]\mathbf{I}_\mathbf{h} \\
&= \begin{pmatrix} \widetilde{y}_{11}^{-1} & \widetilde{y}_{12}^{-1} & \cdots \\ \widetilde{y}_{21}^{-1} & \widetilde{y}_{22}^{-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} e^{j\alpha_1} & 0 & \cdots \\ 0 & e^{j\alpha_2} & \\ \vdots & & \ddots \end{pmatrix} \mathbf{I}_\mathbf{h} \\
&= \begin{pmatrix} \widetilde{y}_{11}^{-1} e^{j\alpha_1} & \widetilde{y}_{12}^{-1} e^{j\alpha_2} & \cdots & \widetilde{y}_{1n}^{-1} e^{j\alpha_n} \\ \widetilde{y}_{21}^{-1} e^{j\alpha_1} & \widetilde{y}_{22}^{-1} e^{j\alpha_2} & \cdots & \widetilde{y}_{2n}^{-1} e^{j\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{y}_{n1}^{-1} e^{j\alpha_1} & \widetilde{y}_{n2}^{-1} e^{j\alpha_2} & \cdots & \widetilde{y}_{nn}^{-1} e^{j\alpha_n} \end{pmatrix} \begin{pmatrix} I_{h1} \\ I_{h2} \\ \vdots \\ I_{hn} \end{pmatrix} \\
&= \begin{pmatrix} \sum_k^n \widetilde{y}_{1k}^{-1} e^{j\alpha_k} I_{hk} \\ \sum_k^n \widetilde{y}_{2k}^{-1} e^{j\alpha_k} I_{hk} \\ \vdots \\ \sum_k^n \widetilde{y}_{nk}^{-1} e^{j\alpha_k} I_{hk} \end{pmatrix}
\end{aligned} \tag{3.13}$$

Hence the voltage at busbar m is given by

$$\begin{aligned}
\widetilde{\mathrm{V}}_{\mathbf{hm}} &= \textstyle\sum_k^n \widetilde{y}_{mk}^{-1} e^{j\alpha_k} I_{hk} \\
&= \textstyle\sum_k^n y_{mk}^{-1} e^{j\beta_{mk}^{-1}} e^{j\alpha_k} I_{hk} \\
&= \textstyle\sum_k^n y_{mk}^{-1} e^{j(\beta_{mk}^{-1}+\alpha_k)} I_{hk} \\
&= \textstyle\sum_k^n y_{mk}^{-1} I_{hk} \cos(\beta_{mk}^{-1} + \alpha_k) + j \sum_k^n y_{mk}^{-1} I_{hk} \sin(\beta_{mk}^{-1} + \alpha_k)
\end{aligned} \tag{3.14}$$

$$\Rightarrow \tan(\theta_m) = \frac{\sum_k^n y_{mk}^{-1} I_{hk} \sin(\beta_{mk}^{-1} + \alpha_k)}{\sum_k^n y_{mk}^{-1} I_{hk} \cos(\beta_{mk}^{-1} + \alpha_k)} \tag{3.15}$$

It can be seen that the prices for injected harmonics given in 3.10, will be dependent on the elements of the inverse admittance matrix.

By making some assumptions about the network it is possible to simplify equation 3.15. First assume all the branch admittances have an angle which is approximately equal ($\beta_{mk} = \beta \approx$ constant, $\forall m \neq k$; this assumption is used by Schweppe *et al* [Schweppe *et al.*1988] in their DC load flow). Then given the way the inverse admittance matrix is constructed each term in $[Y_h]^{-1}$ will have an angle that is approximately equal, or zero magnitude ($\beta_{mk}^{-1} = \beta^{-1} \approx$ const $\forall \;\; y_{m,k}^{-1} \neq 0$). Then if all the non-linear loads inject harmonics at a common angle ($\alpha_k = \alpha =$ constant) one is left with the following expression for voltage angle

$$\begin{aligned}
\tan(\theta_m) &= \frac{\sin(\beta^{-1} + \alpha) \sum_k^n y_{mk}^{-1} I_{hk}}{\cos(\beta^{-1} + \alpha) \sum_k^n y_{mk}^{-1} I_{hk}} \\
\therefore \tan(\theta_m) &= \tan(\beta^{-1} + \alpha) \\
\theta_m &= \beta^{-1} + \alpha
\end{aligned} \tag{3.16}$$

So that under these assumptions the diagonal matrix of voltage angles $[e^{j\theta_h}]$ is given by

$$[e^{j\theta}] = \begin{pmatrix} e^{j(\beta^{-1}+\alpha)} & 0 & \cdots & \\ 0 & e^{j(\beta^{-1}+\alpha)} & & \\ \vdots & & \ddots & \\ & & & e^{j(\beta^{-1}+\alpha)} \end{pmatrix} \tag{3.17}$$

Therefore the resulting prices given by equation 3.10 will all have an angle $\approx -\alpha$, and the marginal price for some injection at busbar m will be that shown in equation 3.18.

$$\widetilde{\mu}_{hm} = e^{-j\alpha} \sum_{t=1}^n y_{tm}^{-1} k_t \tag{3.18}$$

It should be noted that should a filter or infinite busbar appear in the system so that the shunt admittance at that busbar was similar to that of the branch admittances then equation 3.18 may not be the resultant price for that busbar. Instead the price at that busbar will be zero. Intuitively this makes sense as at an infinite busbar no harmonic voltage can exist, therefore the value of any currents injected into that busbar will be zero. But in the case of non-infinite busbars throughout the network, the payments made to each harmonic source will be given by:

$$\begin{aligned}\widetilde{\mu}_{hm}\widetilde{I}_{hm} &= I_{hm}\sum_{t=1}^{n} y_{tm}^{-1}k_t \\ &= I_{hm}\mu_{hm}\end{aligned} \tag{3.19}$$

Under the assumptions, the marginal pricing of the harmonics has the nice property that each load will be charged a real amount for their injections. The restriction that each load can only inject at a common angle is rather strong and unrealistic. When this is relaxed, loads will be charged complex amounts for their injections. It is shown in Chapter 4 that it is only the real part of the amount charged to any load, that is of importance. The complex parts of the amounts charged, is linked to injections by the sources, which are quadrature to the prevailing harmonic voltage. All the complex payments cancel each other out across a whole network, as equation 3.11 indicates the total payments must sum to a real number.

To this point the assumption has been made that all loads inject harmonics at the same angle. Relaxing this assumption slightly it is possible to consider the payments of loads, which inject harmonics with a π radian phase shift, ie. loads which absorb harmonic current from the network. Should there be a minority of loads which inject into the network with a phase shift of π radians, the prices given in equation 3.10 will still be valid as the prevailing harmonic voltage angles through out the network will be unchanged (though the magnitudes will not). Should such a load exist at busbar k:

$$\widetilde{I}_{hk} = I_{hk}e^{j(\alpha+\pi)} \tag{3.20}$$

The the amount charged to such a load which is sucking harmonic current from the network is given by:

$$\begin{aligned}\widetilde{\mu}_{hk}\widetilde{I}_{hk} &= I_{hk}e^{j(\alpha+\pi)}e^{-j\alpha}\sum_{t=1}^{n} y_{tk}^{-1}k_t \\ &= -I_{hk}\sum_{t=1}^{n} y_{tk}^{-1}k_t \\ &= -I_{hk}\mu_{hk}\end{aligned} \tag{3.21}$$

Comparison of equation 3.21 with 3.19, indicates that a load which sucks the prevailing harmonic current from the system faces the opposites charges of the other loads, or in other words they will get paid for their injections. This makes sense in that their injections are in fact reducing the prevailing voltages through out the network, and hence at the margin improving the utility of all connected to the network. This treatment of those making beneficial injections into the network is not universally accepted, with Emanuel of the belief that all injections into the network should be treated as harmful and hence charged a positive amount [Emanuel1999]. I disagree with this view, as efficiency can never be achieved unless each party faces the marginal costs of their actions. If at the margin a particular action is of benefit it should be encouraged, as is shown to be the case in equation 3.21.

3.3.1 Examples of $\widetilde{\mu}_h$

This section contains examples of marginal prices based on the test system detailed in Appendix A. Along with the development of the prices, the amount of money paid and received by each load is calculated, and the results of changing the valuation of harmonic distortion ($\frac{\partial u_i(V_{hi})}{\partial V_{hi}}$ for some i), and harmonic injections ($\widetilde{I}_{hi}$) by loads, is investigated. The resultant harmonic voltages and harmonic prices for each of the busbars is given in Table 3.1 and Figures 3.1, 3.2, and 3.3. It should be noted the cardinal value of all prices and payments in this thesis are meaningless, they have comparitive vaue only. This is on account of the fact the marginal valuation of distortion for each load (k_i), as described in Appendix A are just arbitrary numbers. As mentioned in section 2.2, there has been little effort to this point, in finding realistic distortion valuations.

Table 3.1 Result Harmonic Voltages and Prices for Base Case Test System

Busbar	Harmonic Voltage $\widetilde{\mathbf{V}}_{\mathbf{h}}$ (pu)	Harmonic Injection Price $\widetilde{\mu}_h$ (\$/pu)
1	$-0.062 + j0.051$	$9.10 + j34.27$
2	$-0.061 + j0.052$	$9.60 + j34.94$
3	$-0.062 + j0.051$	$9.12 + j34.24$
4	$-0.062 + j0.051$	$9.21 + j34.17$
5	$-0.061 + j0.052$	$9.47 + j34.02$
6	$-0.062 + j0.051$	$9.11 + j34.22$
7	$-0.061 + j0.052$	$9.42 + j34.06$
8	$-0.062 + j0.051$	$9.14 + j34.26$
9	$-0.062 + j0.051$	$9.06 + j34.38$

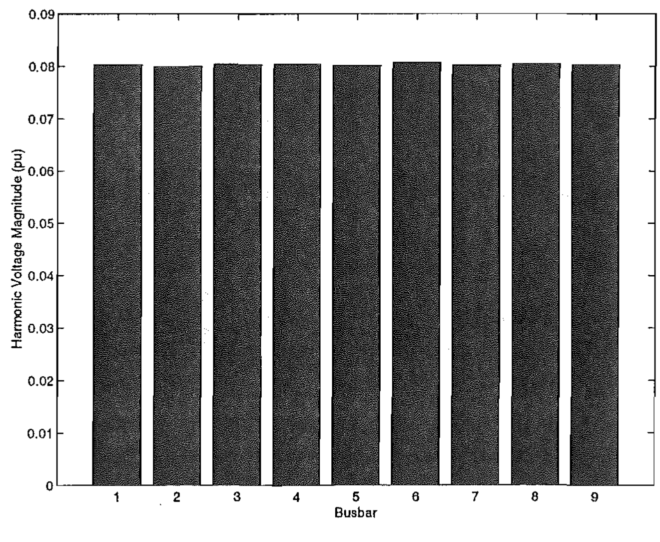


Figure 3.1 Harmonic voltage magnitude at each busbar

Of note here is that the harmonic voltage magnitudes and prices, are approximately equal for each busbar. This is as would be expected for a small strong network such as our test system. In such a case where the injections at any one busbar are likely to have an equal effect on all other busbars throughout the network, one would expect the prices and voltages to be roughly equal throughout the network. In equation 3.18 it was shown that the price for harmonic current injection at any busbar should have the opposite phase angle of the injection itself. This is shown to be the case in Table 3.1. The result being that amounts charged to each of the loads for their injections is a real number, despite the fact that both the injections and prices have a complex form. The payments made to the loads, by the loads and the net payments by the loads are shown in Figures 3.4, 3.5 and 3.6.

The payments made to the loads is the compensation the loads receive for the voltage distortion seen at their busbar and these payments are given by

$$\text{Payment made to the load at busbar}\, i = -k_i V_{hi} \tag{3.22}$$

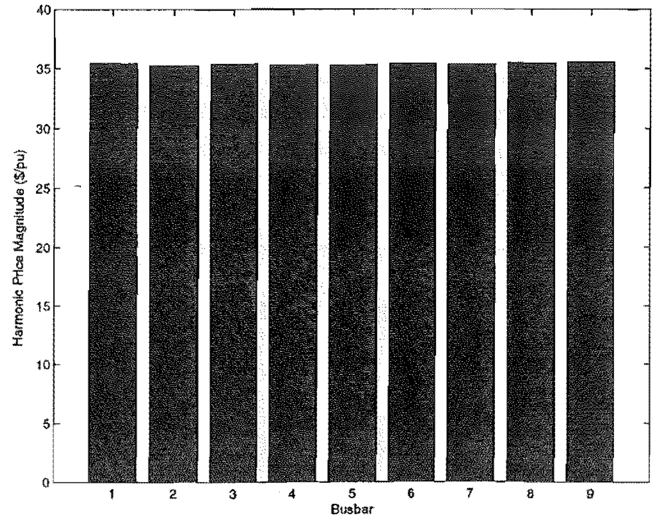


Figure 3.2 Magnitude of marginal price for harmonic injections

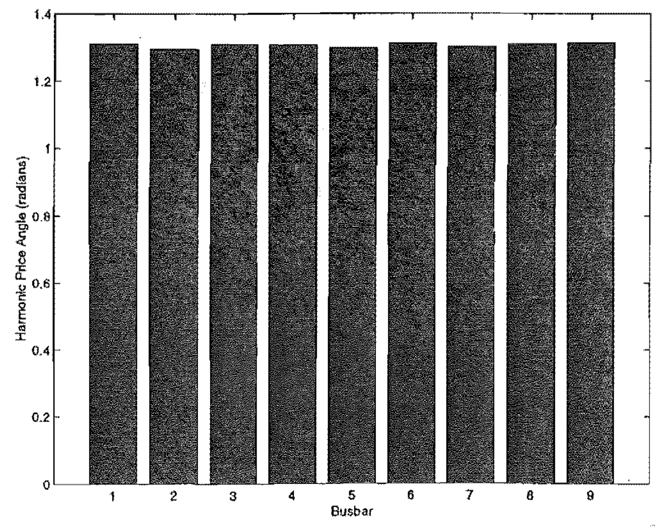


Figure 3.3 Angle of marginal prices for harmonic injections

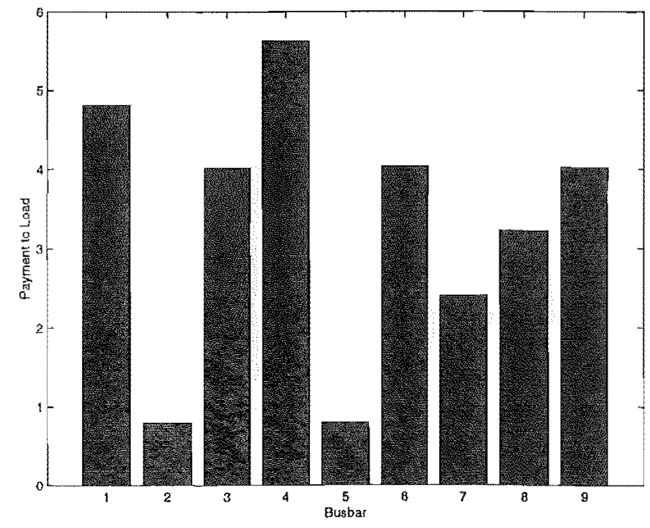


Figure 3.4 Harmonic payments made to each load as compensation for distortion at local busbar

The payments given by equation 3.22 are based on the harmonic property rights being allocated so that loads have the right to a clean voltage supply at their busbar. This need not be the case and other possible allocation of harmonic property rights are detailed in section 3.5. Given that the harmonic voltage seen at each busbar is the same throughout the network, the different payments to each load reflect the different valuation the loads place on voltage distortion (ie. the magnitude of the elements of $\mathbb{K}$).

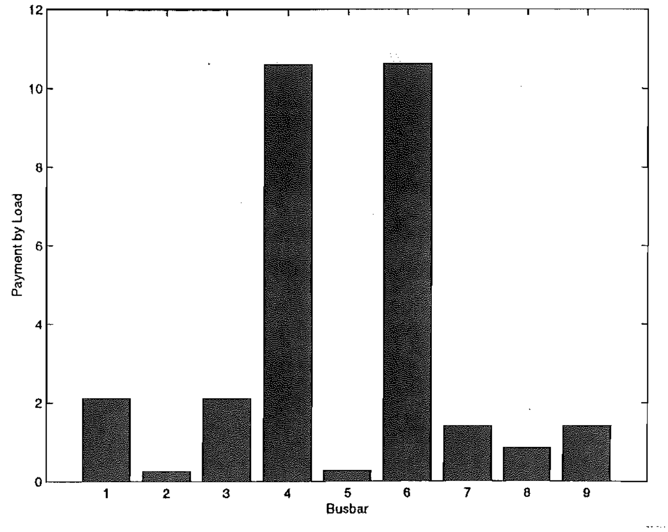


Figure 3.5 Harmonic payments made by each load for current injections

As the price for harmonic injections is roughly equal throughout the network, the amount charged to each load reflects the magnitude of their injections into the network (in the case where all loads inject at a common angle). The result being, in this example (3.5) the loads at busbars four and six end up paying the majority of charges, as they account for the majority of injections (detailed in Appendix A).

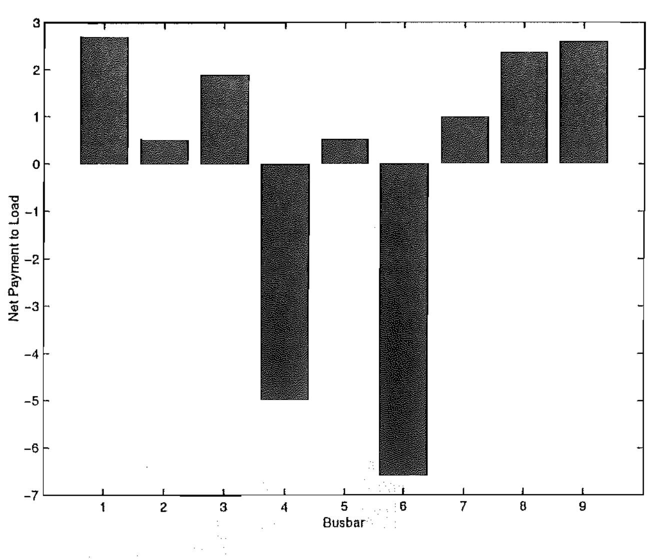


Figure 3.6 Net payment made to the loads at each busbar

Figure 3.6 shows the net payments made to each load. This is of interest as it shows as result of the marginal pricing system, is the loads at busbars four and six, end up compensating the other loads for the distortion they cause. That is not to say that the other loads do not cause voltage distortion (they do) or that the polluting loads are not compensated for the distortion they see (they are). But in this test system there are two loads causing the majority of all distortion seen in the system, and hence under marginal pricing they bear the majority of the harmonic costs this distortion creates through out the network. This characteristic of each load seeing the true cost their actions have on the whole network, is an important one with respect to achieving economic efficiency.

Having established what the prices and the payments between the loads look like to the base case test system, Figues 3.7, 3.8, 3.9 and 3.10, show how the harmonic prices and payments between loads vary when the load at busbar four has a marginal valuation of voltage distortion that varies between 0 and -350 \$/pu.

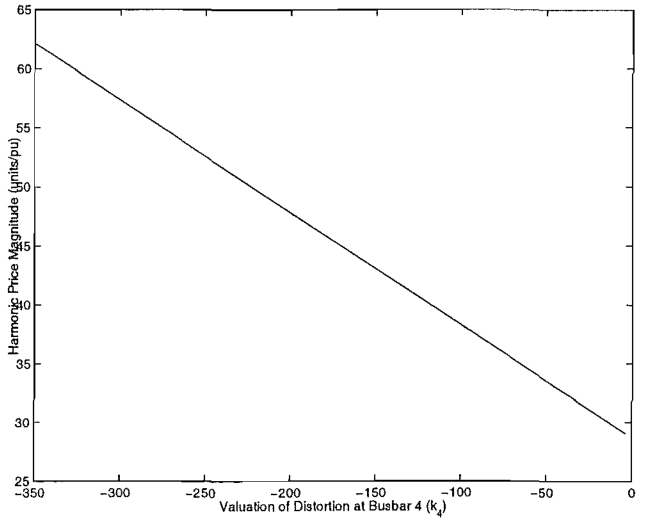


Figure 3.7 Magnitude of harmonic prices as k_4 is varied from 0 to -350

Figure 3.7, shows that the mean price magnitude for harmonic injections through out the network. The price at each busbar will be approximately equal to this value, as the test system is a strong network. As k_4 increases (in magnitude) there is a linear increase in the price magnitude, this is as expected given that under these simplified conditions the price is a linear function of the different marginal distortion valuations as shown in equation 3.18.

Figure 3.8 shows the distortion compensation payments received by each load as k_4 changes. The harmonic current injections into the network are not changing and hence neither is the harmonic voltages seen at each busbar. The result being that the compensation received by each load is constant except for the load at bus four, which has a varying valuation of voltage distortion. As the load at busbar four increases its valuation of harmonic distortion (k_4) it is paid more compensation. This naturally leads to the question of weather loads are going to have incentives to misrepresent their valuation of distortion in an effort to extract extra payments, this question is dealt with in Section 3.6.

Figure 3.9, shows the payments made by each load for their harmonic injections. As a result of the marginal prices being a linear function of k_4, the payments made by each load are also a linear function of k_4, as the actual injections made by each load are not changing.

Finally figure 3.10 shows the net payments received by each load as k_4 is varied. Again the net payments received by each load will be a linear function of k_4, as both what they are charged for their injections and what they receive in compensation for voltage distortion are linear functions of k_4. Of interest here is how as the costs of voltage distortion increase for the load at busbar four, that load goes from being harmonic cashflow negative to harmonic cashflow positive. This is as when $k_4 = -350$, over half all the value the network places on harmonic distortion is attributable to the load at busbar four. So despite that the load at busbar four, is

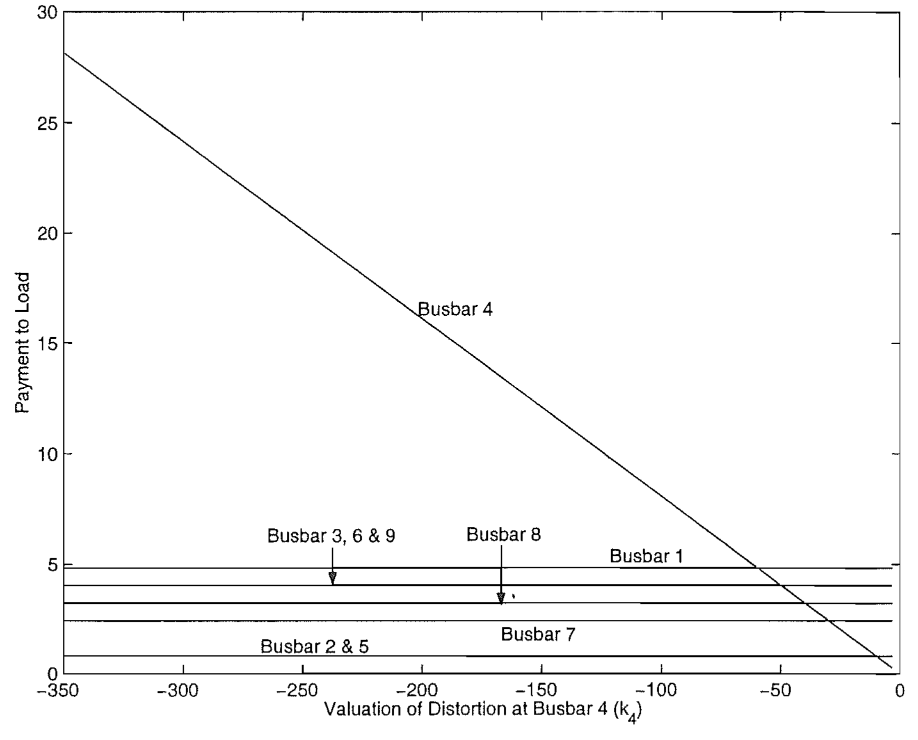


Figure 3.8 Payments made to the loads at each busbar as k_4 is varied from 0 to -350

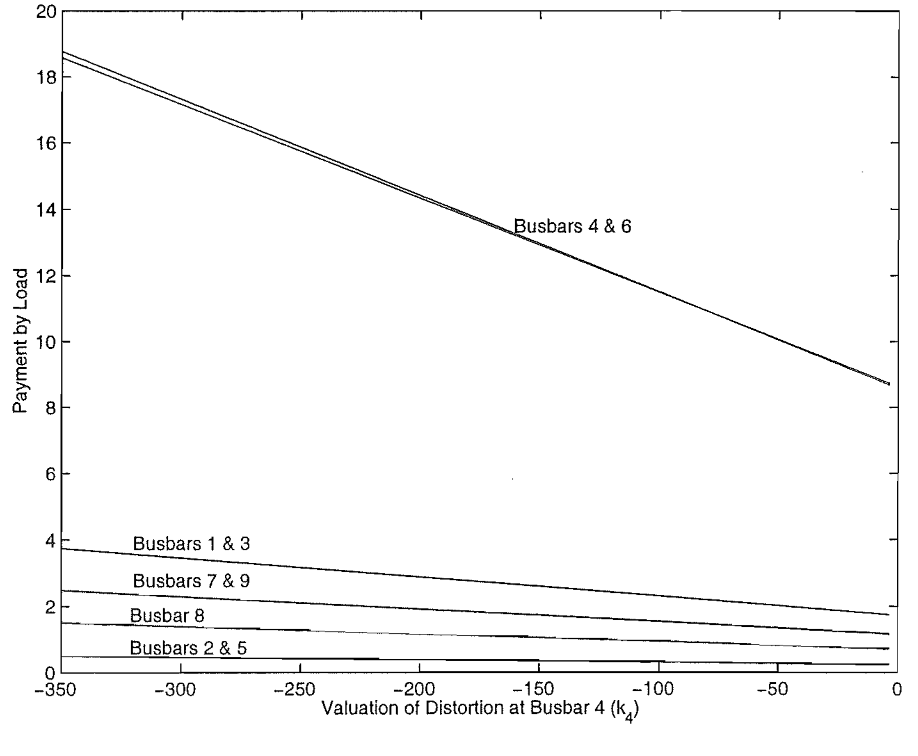


Figure 3.9 Payments made by the loads at each busbar as k_4 is varied from 0 to -350

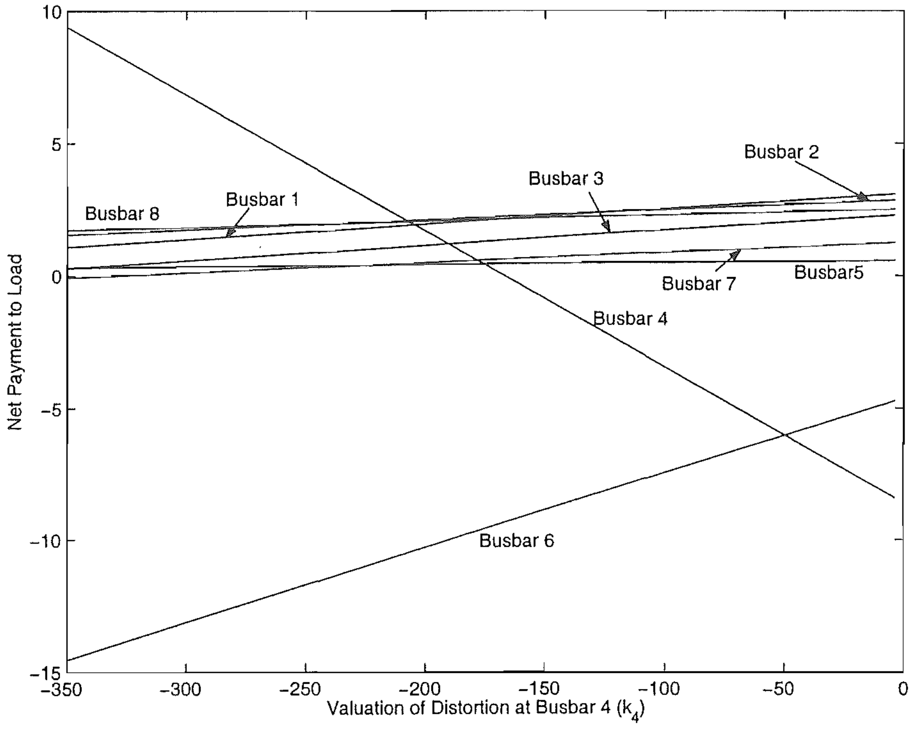


Figure 3.10 Net payment made to the loads at each busbar as k_4 is varied from 0 to -350

responsible for close to half the distortion, it is still paid.

Also noteworthy is how these elementary prices behave in the presence of a load whose harmonic injections into the network are not constant. In figures 3.11, 3.12, 3.13, 3.14 and 3.15, the behaviour of the harmonic voltages, prices and payments between loads is demonstrated as the harmonic current injections for the load at busbar four is varied from 0 to 1.5 pu.

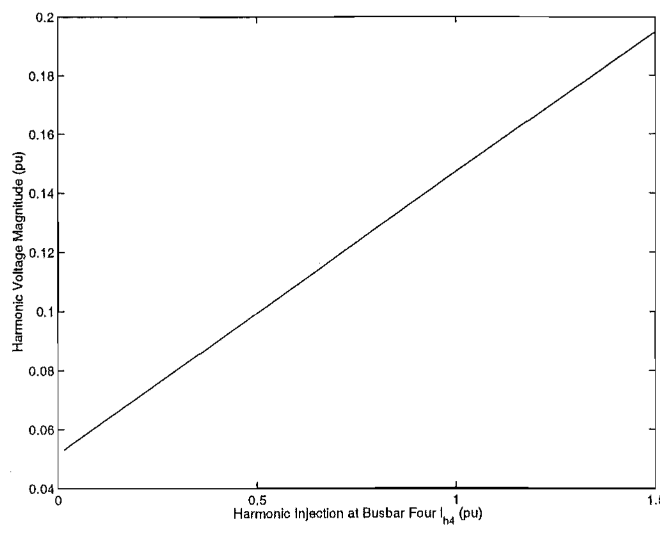


Figure 3.11 Harmonic voltage magnitude throughout the network as I_{h4} is varied from 0 to 1.5 pu

Figure 3.11, shows how the prevailing voltage magnitude behaves as I_{h4}, is increased. No great insights here considering that the voltages through out the network are a linear function of the current injections as shown in equation 3.23.

$$\widetilde{\mathbf{V}}_\mathbf{h} = [Y_h]^{-1}\widetilde{\mathbf{I}}_\mathbf{h} \tag{3.23}$$

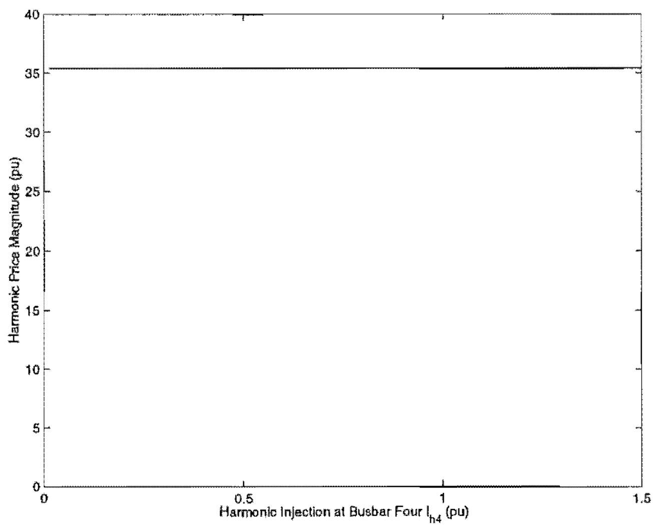


Figure 3.12 Harmonic price magnitude throughout the network as I_{h4} is varied from 0 to 1.5 pu

Figure 3.12 demonstrates an interesting property which exists when the cost of harmonic distortion to loads is a linear function of the the distortion magnitude as assumed here. As shown in equations 3.18 and 3.10, the marginal price for harmonic injections is independent of the voltage magnitude where $\frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} = \mathrm{K}$. This makes sense in that should the costs to loads be a linear function of the voltage distortion, the aggregate marginal value to the network of any injected harmonic, is independent of the harmonic state of the network, and constant prices reflects this.

The under assumptions of this example the payments made to each load as compensation for the voltage distortion, is a linear function of the voltage magnitude (this is shown in equation 3.22). Given the linear increase in prevailing harmonic voltages with I_{h4}, one would also expect to see a linear increase in the compensation payments each load receives as is seen to be the case in figure 3.13.

Figure 3.14 shows how the only payments made by the load at busbar four vary. This makes sense in that the marginal value of injected current is constant (as described by the prices) and only the load at busbar four is changing their level of injected harmonic current.

Figure 3.15 shows how when the load at busbar four, increases its injections the result is increasing net payments being made to all the other loads in the network due to the effect that action has on these loads. As such the load at busbar four, clearly sees and bears the cost of his increased injections. This leaves the load at busbar four, in a position to make a rational decision based on their valuation of the increased injection versus the compensation payments it will need to make to the other network participants. Under marginal pricing all the other loads are indifferent to the actions of the loads at busbar four, as they are compensated to the extent of any costs they face due to his actions.

3.4 OPTIMISATION PROPERTIES

The two examples in the Section 3.3.1 showed how when the value of harmonic distortion to the network changed, this change was transmitted to each member of the network via the prices

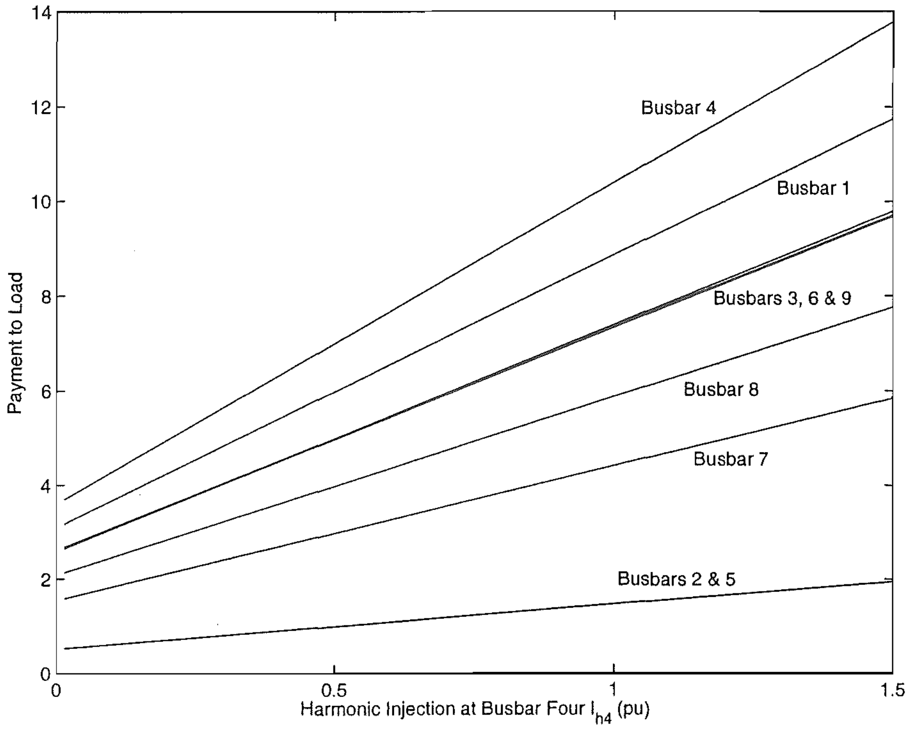


Figure 3.13 Payments made to the loads as compensation for voltage distortion, as I_{h4} is varied from 0 to 1.5 pu

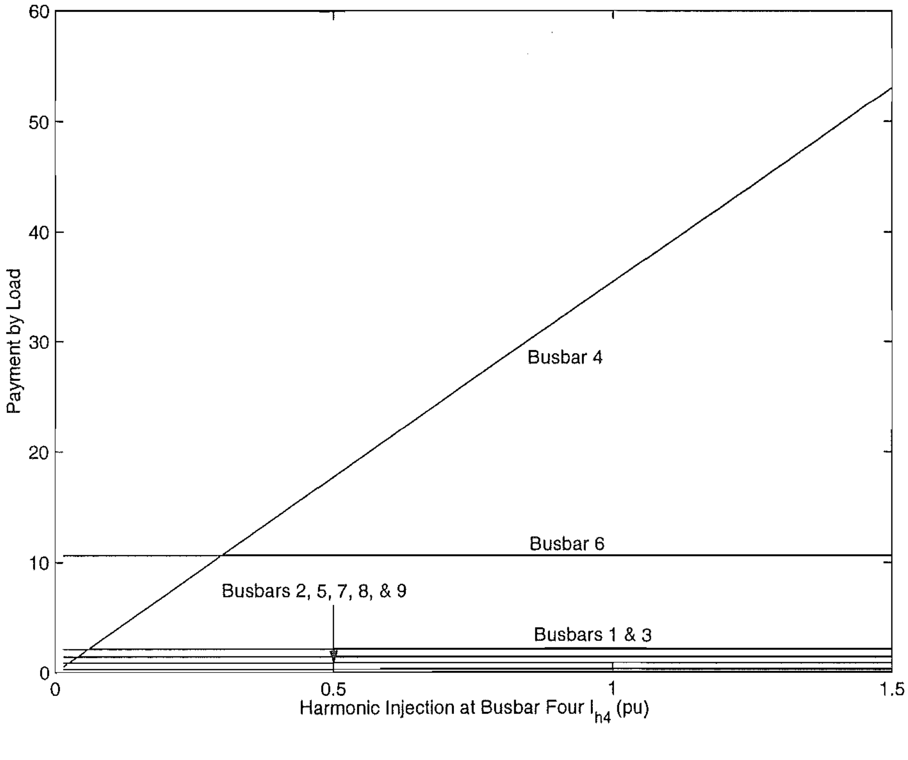


Figure 3.14 Payments made by the loads for harmonic current injections, as I_{h4} is varied from 0 to 1.5 pu

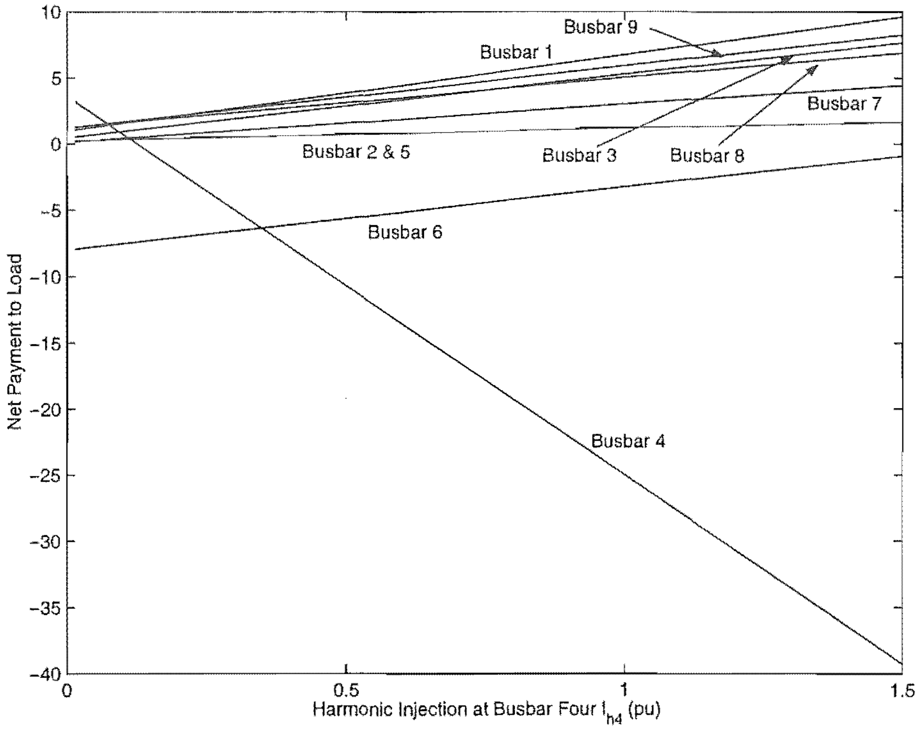


Figure 3.15 Net payments made to the loads, as I_{h4} is varied from 0 to 1.5 pu

they face for their injections. While on the other hand each load was indifferent any action a given member of the network may take so long as it does not effect the marginal value of their injections. Hence the previous examples highlighted how marginal pricing has the ability to coordinate the activities of individuals so that optimality is achieved. A proof of this is demonstrated here. When developing the prices earlier the loads were considered inert, in that their actions were not influenced by the harmonic prices. Should this be the case in reality there would be no point in developing the prices (apart from the fact they would "fairly" allocate the costs the harmonics incurred on others). So the question is when one allows the loads to respond to prices do they act in an optimal fashion?

Consider that loads have the have the ability to reduce their injections into the system at a cost dependent on the magnitude of the current reduction (I_R).

$$\text{Harmonic reduction cost for busbar}\, i = RC_i(I_{Ri}) \tag{3.24}$$

So that the total cost to the system due to the reduction of harmonic injections will be a function of the vector of current reductions ($\mathbf{I_R}$).

$$\text{Total harmonic reduction cost} = RC(\mathbf{I_R}) \tag{3.25}$$

There are two constraints on $\mathbf{I_R}$

$$\mathbf{I_R} \geq \mathbf{0} \tag{3.26}$$
$$\mathbf{I_R} \leq \mathbf{I_h} \tag{3.27}$$

Equation 3.26 reflects that $\mathbf{I_R}$ is a magnitude vector and hence cannot be negative. If $\mathbf{I_R}$ were to go negative this would be the equivalent of the load taking action to inject more harmonic current into the system, it's assumed this is not possible. Equation 3.27 implies that any load can only reduce their harmonic output to zero, they can not inject the harmonic with a phase shift of π radians. That is no load will install excess harmonic reduction capacity.

To show that harmonic pricing will lead to an efficient out come the optimum must be found. The Lagrangian for this new model is:

$$\begin{aligned}\mathcal{L}(\mathbf{V_h}, \mathbf{I_R}, \widetilde{\mu}_h, \lambda_1, \lambda_2) =& U(\mathbf{V_h}) - RC(\mathbf{I_R}) + \widetilde{\mu}_h(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}) \\ &+ \lambda_1(\mathbf{I_R}) + \lambda_2(\mathbf{I_h} - \mathbf{I_R})\end{aligned} \tag{3.28}$$

The first order conditions which describe optimality for this revised problem are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} - \widetilde{\mu}_h[Y_h]\frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} &= \mathbf{0} \\ &\text{(Identical to equation 3.5)}\end{aligned} \tag{3.29}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{I_R}} &= -\frac{\partial RC(\mathbf{I_R})}{\partial \mathbf{I_R}} - \widetilde{\mu}_h\frac{\partial \widetilde{\mathbf{I_R}}}{\partial \mathbf{I_R}} + \lambda_1 - \lambda_2 = \mathbf{0} \\ \Rightarrow \frac{\partial RC(\mathbf{I_R})}{\partial \mathbf{I_R}} &= -\widetilde{\mu}_h[e^{j\alpha}] + \lambda_1 - \lambda_2\end{aligned} \tag{3.30}$$

Equation 3.30 states that in the absence of any constraints each load should continue to reduce their harmonic injections to the point where the cost of doing so equals what that load is effectively paid for the reduction. In the case where harmonic injections into the network impose large costs on others ($\Rightarrow \mu_{hi}$ is large) the amount paid for harmonic reductions (μ_{hi}) may always exceed the marginal cost($\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}}$), in this case $\boldsymbol{\lambda_2} > \mathbf{0}$. Similarly where the marginal cost of reducing the harmonics always exceeds the value of doing so $\boldsymbol{\lambda_1} > \mathbf{0}$. These results intuitively make sense, reduce your injections until it is no longer worth your while or until they have been reduced to zero.

Equation 3.30 states the optimality condition for the system as a whole. This condition in terms of each load can be expressed as:

$$\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} = -\widetilde{\mu}_{hi}e^{j\alpha} + \lambda_{1i} - \lambda_{2i} \tag{3.31}$$

As can be seen equation 3.29 is identical to equation 3.5. This suggests that the optimal prices to be used are the same as those before. Its is worth noting that no knowledge of the loads' reduction costs are required in calculating these prices. The problem then faced by the load is:

$$\text{Maximise} \quad \widetilde{\mu}_{hi}(\widetilde{I}_{hi} - \widetilde{I}_{Ri}) - RC_i(I_{Ri}) \tag{3.32}$$

$$\text{Subject to} \quad I_{Ri} \geq 0 \tag{3.33}$$

$$I_{Ri} \leq I_{hi} \tag{3.34}$$

So the Lagrangian for the individual load is

$$\mathcal{L}(I_{Ri}, \lambda_{1i}, \lambda_{2i}) = \widetilde{\mu}_{hi}(\widetilde{I}_{hi} - \widetilde{I}_{Ri}) - RC_i(I_{Ri}) + \lambda_{1i}(I_{Ri}) - \lambda_{2i}(I_{hi} - I_{Ri}) \tag{3.35}$$

Which has the following first order condition.

$$\frac{\partial \mathcal{L}}{\partial I_{Ri}} = -\mu_{hi}e^{j\alpha} - \frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} + \lambda_{1i} - \lambda_{2i} = 0 \tag{3.36}$$

Equation 3.36 shows the same condition as equation 3.31. The Lagrange multipliers take on the same value, as the constraints faced are identical along with the cost function and harmonic injection price. This shows that the marginal prices, which are calculated using just voltage measurements and knowledge of individual preferences, can be used to coordinate harmonic reduction amongst the loads. In this section it was assumed that each load is a price taker (has no or little ability to influence the price). This price taker assumption is valid where there are many loads in the network, or where the utility of each load is a linear function of the harmonic voltage distortion seen at their busbar (as in equation 3.9).

3.5 MARKET IMPLEMENTATION AND PROPERTY RIGHTS

By using marginal pricing to convey the value of harmonic injections to each network participant it is possible coordinate actions so optimality is achieved. This opens up the theoretical possibility of using a harmonic market to manage distortion levels and compensate loads (as done previously in section 3.3.1). But before harmonic prices can be used as the basis for a harmonic market, harmonic property rights must first be established. There are a number of different ways the market for harmonic injections can operate depending on how one wants to allocate harmonic property rights. There are two basic methods by which harmonic property rights can be allocated:

1. Every load has a right to a harmonic free supply. Nonlinear loads will be charged for their harmonic injections and these payments will be passed on to the effected loads
2. Every load has the right to inject what it likes into the system. Effected loads have the right to pay other loads to reduce their harmonic inputs.

It is also possible to get an allocation anywhere in between these two allocations by allocating pollution rights, which are then tradeable. Of interest is what role pricing of harmonic injections has under these two different allocations and how the payments among system participants differ as the property rights differ.

3.5.1 Right to a Clean Supply

Under this system every load has the right to charge for any voltage distortion seen at their busbar. These charges form system utility function $U(\mathbf{V_h})$, in the case where each load has a constant marginal cost associated with distortion this function is given by $U(\mathbf{V_h}) = \mathbf{KV_h}$. This system utility function is used to generate the optimal prices for harmonic injections (given in equation 3.10).

To implement this system a market mechanism must be set up which will process the harmonic charges (develop $U(\mathbf{V_h})$); calculate the appropriate prices ($\widetilde{\mu}_h$); charge the offending loads the appropriate amount; and then compensate each load i according to $u_i(V_{hi})$, as described in equation 3.22. Under this system, loads have two options. They can do nothing to reduce their harmonic injections, an option which will be taken when $\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} > \mu_{hi}$. Or should $\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} < \mu_{hi}$ the load will choose to reduce their harmonic input until either $\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} = \mu_{hi}$, or until harmonic injections are reduced to zero. These actions on behalf of the nonlinear loads match the optimality conditions in 3.31. Hence this allocation of property rights in conjunction with the developed harmonic prices will lead to an optimal outcome.

3.5.2 Right to Pollute

Or one could give each load the right to inject what they liked into the system. Under this situation it may appear that there would be no need for a formal pricing system to be set up, as no loads are charged for their injections. Though each load has the right to pay other loads to reduce their harmonic injections. It is shown in Appendix B, that in the absence of an organised market if each load is left on an individual basis to decide what to pay the other loads the result will be non-optimal. This is as the harmonic injections have similar properties to that of a public good. As such should each load be given the right to pollute, then it will still be desirable to have an organised market in which harmonic prices are explicitly calculated.

With an Organised Market

A formal market mechanism to establish prices, allows individual's actions to be coordinated and optimality achieved. When loads are given the right to pollute a market mechanism similar to that in section 3.5.1 must be developed. Again all market participants would submit their valuation of harmonic voltage distortion, which would be used to develop the system utility function $U(\mathbf{V_h})$. This is used to develop prices for the injections. As shown in equation 3.18, should the loads have a fixed marginal utility of harmonic distortion and the harmonic currents be injected with a common angle (or π radians out of phase), the shadow price at a busbar s, is given by

$$\mu_{hs} = \sum_{j}^{n} k_j y_{js}^{-1} \tag{3.37}$$

This price is known to represent the system's marginal valuation of the injected harmonic at busbar s. Hence 3.37 is what the system as a whole is willing to pay the load s, to reduce their harmonic output. This payment to load at s, would then be split amongst all the other loads according to

$$\text{Payment by load z} = \frac{k_z y_{zs}^{-1}}{\sum_j^n k_j y_{js}^{-1}} \mu_{hs} I_{rs} \tag{3.38}$$

Where equation 3.38 is easily simplified to:

$$\text{Payment by load z} = k_z y_{zs}^{-1} I_{rs} \tag{3.39}$$

The marginal cost for the load at s reducing their harmonic injections, compared to μ_{hs} will determine whether or not s reduces their harmonic injections. This is in contrast to the case where there is no organised market and the level of harmonic reduction is determined by the relationship of ρ_s and the largest value of $\left(\frac{\partial u_t}{\partial V_{ht}}\right) y_{ts}^{-1} \quad \forall t$.

While the above section has detailed how to encourage efficient behaviour from the load at s (given it's right to pollute), there is no reason why any individual could not install mitigation equipment at busbar s, and receive the payment μ_{hs}. As in section 3.5.1 an optimal harmonic profile should be achieved.

3.5.3 Comparison of Property Rights

It has been shown in Section 3.4 that marginal pricing should provide incentives to act in an optimal fashion, and hence minimise the total costs to the system of harmonic distortion. It was also shown in Section 3.5 that there are multiple ways in which marginal pricing can be implemented depending on how the harmonic property rights are allocated. In this section using the test system described in Appendix A, marginal prices are calculated, and each load reacts in a rational manner. This rational behaviour was described earlier in equation 3.36. In the case where each load has a right to a clean supply (marginal clean pricing), rational behaviour means that should the price of harmonic injections exceed the cost of reducing injections, the load will reduce their injections to the point where the two are equal, or till injections have been reduced to zero. In the case where each load has the right to pollute (marginal dirty pricing), each load sees what the rest of the network as a whole is willing to pay them to reduce their injections, and should this amount exceed their costs of rejection reduction, they will reduce their injections till the price equals the cost, or they have reduced their injections to zero. Independent of how the property rights are allocated the the decision each load makes with respect to their injections is based on the marginal prices, which represent the marginal value of those injections to the whole network. The consequences of allowing the loads to adjust their injections in a rational manner to the prices they face is shown in Table 3.2.

Table 3.2 Resultant Harmonic Injections and Voltages Before and After Injection Reductions

Busbar	$\mathbf{I_h}$ (pu)	Intial $\mathbf{V_h}$ (pu)	μ_h (\$/pu)	ρ (\$/pu)	$\mathbf{I_h} - \mathbf{I_R}$ (pu)	Final $\mathbf{V_h}$ (pu)
1	0.0600	0.0804	35.46	60	0.0600	0.0461
2	0.0080	0.0800	35.27	100	0.0080	0.0459
3	0.0600	0.0804	35.44	25	**0.0000**	0.0461
4	0.3000	0.0805	35.39	80	0.3000	0.0463
5	0.0080	0.0801	35.32	100	0.0080	0.0459
6	0.3000	0.0807	35.41	15	**0.0000**	0.0456
7	0.0400	0.0802	35.34	42	0.0400	0.0459
8	0.0243	0.0805	35.46	58	0.0243	0.0457
9	0.0400	0.0804	35.56	42	0.0400	0.0457

In this example, as before each load is assumed to have a constant marginal utility that results from harmonic distortion seen at their busbar ($\frac{\partial u_i(V_{hi})}{\partial V_{hi}} = k_i$), and it is also assumed that each load can reduce their injections at a constant marginal cost ($\frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} = \rho_i$). The first thing to note is that $\rho_3 < \mu_{h3}$ and $\rho_6 < \mu_{h6}$. Hence the loads at both busbars three and six have an incentive to reduce their injections to zero, resulting in a reduction in the prevailing harmonic voltage throughout the network. This action from the loads at busbars three and six has consequences for the total harmonic utility/costs of the network. In the case of both marginal clean and dirty pricing the total utility of network participants (resulting from harmonic distortion) is given by equation 3.40. The resultant harmonic utility for the test system from before and after the pricing is implemented is shown in Table 3.3.

The implementation of the marginal pricing scheme, provided an incentive for the loads at busbars three and six to under take action which improved the welfare of the whole network. It is worth noting that the actions, which result from marginal pricing are Pareto efficient. That is once property rights have been assigned either explicitly or implicitly (in the case where no marginal pricing exists), no network participant is worse off from the establishment and trading on the basis of marginal prices; yet the welfare of the network in aggregate is improved. A corollary of this is that any network in which the value of the actions of each individual is not clearly signalled, and in which there is no incentive for loads to move towards efficient injection levels is unlikely to be operating at a Pareto efficient point.

$$\text{Total Distortion Utility} = \mathbf{K V_h} - \rho \mathbf{I_R} \tag{3.40}$$

It is of interest here that both possible allocations of property rights, result in the same action from all loads in terms of harmonic reduction implemented, resulting in the same costs to the network from harmonic distortion. This is a product of the way the load's preferences were constructed. As stated each load has the choice to pay the market price at their busbar for harmonic injections or reduce their injections. Essentially this problem for each load boils

Table 3.3 Total Utility Due to Harmonic Distortion Before and After Marginal Pricing

	Before Marginal Pricing	After Marginal Pricing
Marginal Clean	−29.76	−23.01
Marginal Dirty	−29.76	−23.01

down to a situation where is load is has the option to purchase all other goods (which takes the form of money) or harmonic injections. Money has a nominal price equal to unity, and the utility of every load is assumed to be a linear function of the amount of money received. This was an assumption implicitly made when the utility function for harmonic distortion was given monetary units. On the other hand the utility of harmonic distortion is a decreasing function of the distortion present (in our example a linear function but it need not be). The reason being that each load has quasi-linear utility functions in the harmonic-money space. In this case the final equilibrium harmonic distortion present will be independent of the allocation of property rights [Varian1996].

Having seen that both marginal clean and marginal dirty pricing has the improved the welfare of the network participants, figures 3.16, 3.17 and 3.18 show how the payments made by the loads, to the loads, and the net payments to the loads, vary between the two pricing systems.

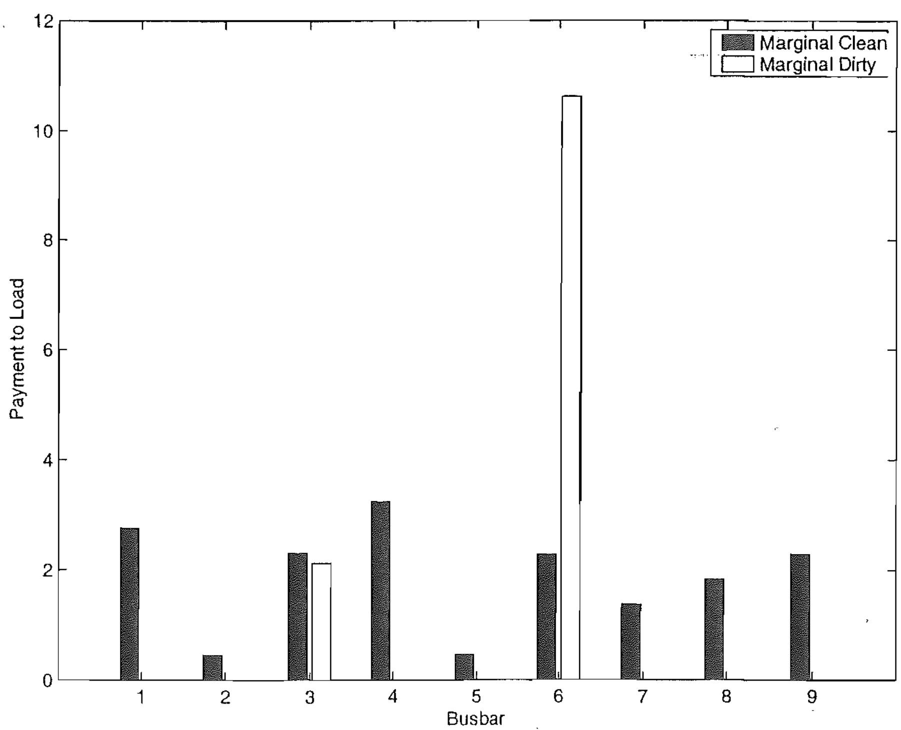


Figure 3.16 Payments made to the loads under marginal clean and marginal dirty pricing

Figure 3.16, shows how there are considerable differences in the payments made to the loads from the two different pricing systems. In the case of marginal clean pricing, the payment made to each load represents their marginal valuation of voltage distortion seen at their busbar, and all loads are paid. On the other hand with marginal dirty pricing only the two loads that reduce their injections are paid. But one has the opposite situation in Figure 3.17, which shows the amount paid by each load. Under marginal dirty pricing the amount paid by each load now represents the marginal valuation of distortion by each load, while the payments under marginal clean pricing are a reflection of each loads injections.

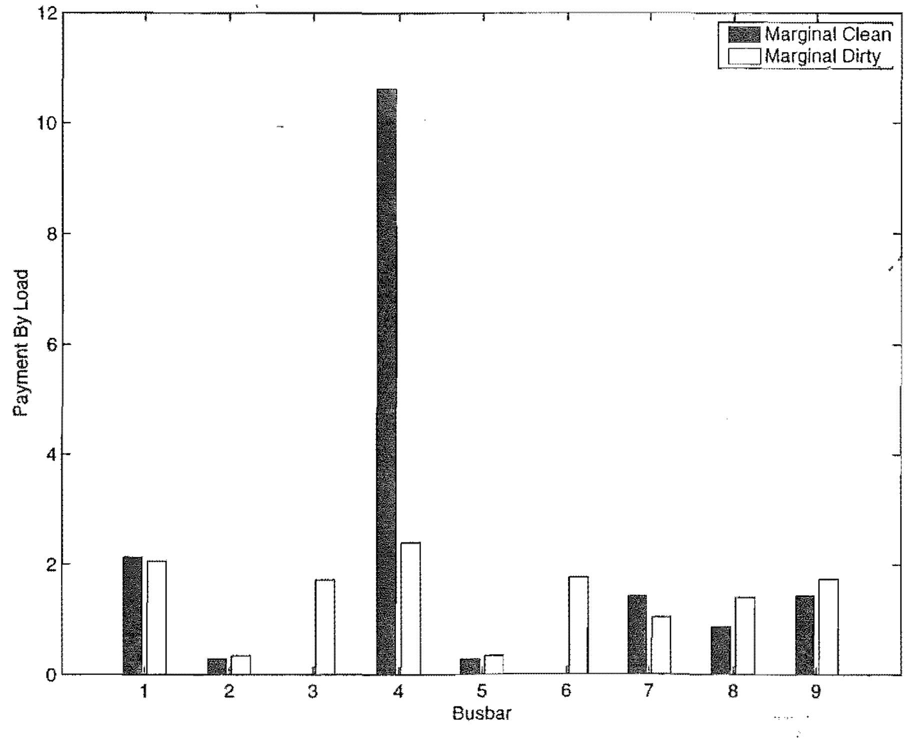


Figure 3.17 Payments made by the loads under marginal clean and marginal dirty pricing

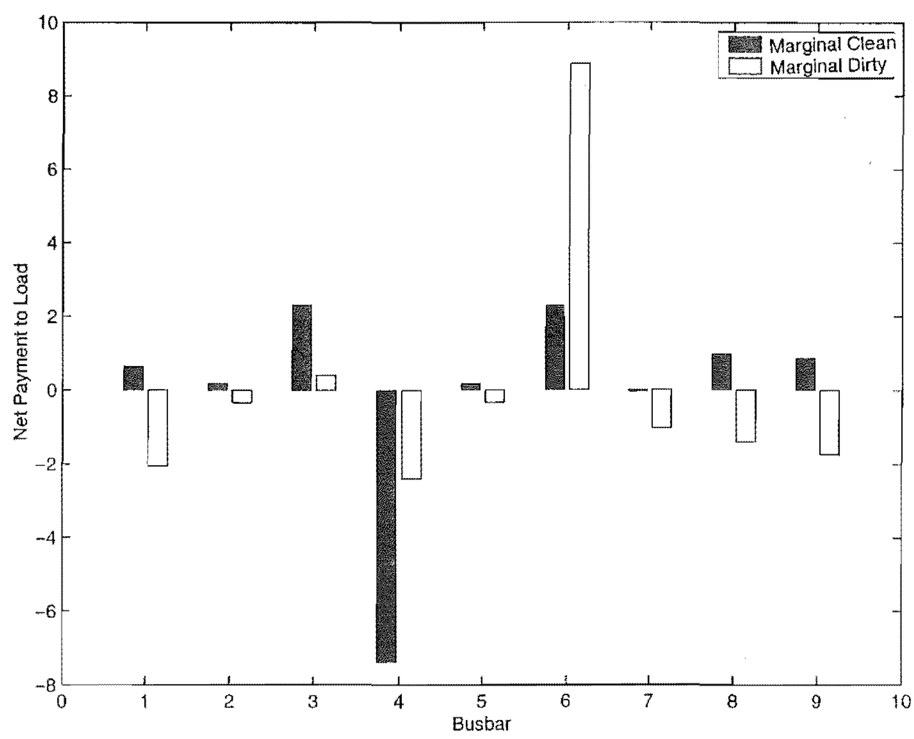


Figure 3.18 Net payments made to the loads under marginal clean and marginal dirty pricing

Figure 3.18, shows that there are considerable differences between the resultant payments made by the loads depending on which pricing system is used. In some cases it is not only the magnitude of the payments which differ but the direction of the payments, meaning a load which under one system had a cash inflow, under the other system may suffer a cash out flow. This would seem to make the previous statement "that once property rights have been assigned, no network participant is worse off from the establishment and trading on the basis of marginal prices" difficult to believe. But this is shown to be the case in Table 3.4, where the utility of each load as a result of harmonic distortion before and after the implementation of both marginal pricing schemes, is shown.

Table 3.4 Load utility under different allocations harmonic property rights

	Marginal Clean		Marginal Dirty	
Busbar	Before Pricing	After Pricing	Before Pricing	After Pricing
1	−2.13	−2.13	−4.82	−4.82
2	−0.28	−0.28	−0.80	−0.80
3	−2.13	−1.50	−4.02	−3.39
4	−10.62	−10.62	−5.63	−5.63
5	−0.28	−0.28	−0.80	−0.80
6	−10.62	−4.50	−4.03	2.09
7	−1.41	−1.41	−2.40	−2.40
8	−0.86	−0.86	−3.22	−3.22
9	−1.42	−1.42	−4.02	−4.02

As can be seen from Table 3.4, it is the actual assignment of property rights which has the large effect on the welfare of each load, especially those that make large harmonic injections into the network, or those which have a very high valuation of voltage distortion seen at their busbar. The specification of marginal prices has only positive effects on the welfare of each load once the property rights are assigned, due to the incentive they provide for efficient action to be taken. The property rights either explicitly stated or not, exist in every network and hence all networks stand to benefit from the calculation of marginal prices. It was mentioned that any system used for allocating harmonic costs should be both efficient and fair. It's been shown that marginal pricing is efficient, Table 3.4 suggests that marginal pricing is also fair, in that once the harmonic property rights have been allocated the loads which reduce their impact on the network are rewarded by an amount equal to the value the network places on the action taken. The cost of this action is shared by all the loads based on their personal valuation of the action.

3.6 LOAD INCENTIVES

The previous sections demonstrated how harmonic prices could be used to coordinate loads so that optimal behaviour on behalf of each is achieved, leading to the possibility of setting up a harmonic market. There are some issues that may not make this initially desirable (apart from the technical metering issues). There was a tacit assumption that the prices were an accurate reflection of the true costs the loads face due to harmonic voltage distortion. In other words, that the loads are truthful when they specify their value of k_i $(= \frac{\partial u_i(V_{hi})}{\partial V_{hi}})$. Should there be an incentive for the load to state a marginal value of harmonic distortion k_i^* that differs from their true value k_i, then the prices will not accurately reflect the value of distortion. These distorted signals to network participants will ultimately result in non-optimal behaviour.

First looking at marginal clean pricing, in considering how any load i will behave with respect to stated k_i^*, how all the other loads behave when facing the prices $\widetilde{\mu}_h$ must first be considered. Assuming that none of the constraints with respect to $\mathbf{I_R}$ will be binding, the result is that each load will reduce their harmonic injections until the marginal costs of doing so equals the marginal payments for the reductions.

$$\frac{\partial RC(\mathbf{I_R})}{\partial \mathbf{I_R}} = -\widetilde{\mu}_h[e^{j\alpha}] \tag{3.41}$$

$$\Rightarrow \frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} = -\widetilde{\mu}_{hi}e^{j\alpha} \tag{3.42}$$

As shown before this behaviour is optimal so long as we have the proper valuation for the harmonic injections (true $\widetilde{\mu}_h$). Note that in the decision making process where each load m, decides how they want to behave when faced with $\widetilde{\mu}_{hm}$, V_{hm} is not a decision variable. This is true for any load which states $k_m^* = k_m$. When this is the case any load is indifferent to voltage distortion that they see at their busbar, as the compensation they receive is equal to their loss of utility.

Now looking at the behaviour of a load which has the option of stating a cost function due to distortion which differs from their true cost function (that is k_i^* need not equal k_i), the problem is to:

$$\text{Maximise} \quad V_{hi}(k_i^* - k_i) + \widetilde{\mu}_{hi}(\widetilde{I}_{hi} - \widetilde{I}_{Ri}) - RC_i(I_{Ri}) \tag{3.43}$$

$$\text{Subject to} \quad \tfrac{\partial RC_{-i}(\mathbf{I_{R-i}})}{\partial \mathbf{I_{R-i}}} + \widetilde{\mu}_{h-i}[e^{j\alpha_{-i}}] = \mathbf{0} \tag{3.44}$$

$$\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h} = \mathbf{0} \tag{3.45}$$

$$\widetilde{\mu}_h - \mathbf{K}^*[e^{j\theta}]^{-1}[Y_h]^{-1} = \mathbf{0} \tag{3.46}$$

where

$$\begin{aligned} RC_{-i}(\mathbf{I_{R-i}}) &= \sum_{k \neq i} RC_k(I_{Rk}) \\ \mathbf{I_{R-i}} &= \text{Vector } \mathbf{I_R} \text{ with element } i \text{ removed} \\ \widetilde{\mu}_{h-i} &= \text{Vector } \widetilde{\mu}_\mathbf{h} \text{ with element } i \text{ removed} \\ \mathbf{K}^* &= (k_1^*, k_2^*, ..., k_n^*) \qquad \text{where} \quad k_m^* = k_m \quad \forall m \neq i \end{aligned}$$

Therefore a load which is looking to optimise k_i^* has the Lagrangian

$$\begin{aligned} \mathcal{L} =& V_{hi}(k_i^* - k_i) + \widetilde{\mu}_{hi}(\widetilde{I}_{hi} - \widetilde{I}_{Ri}) - RC_i(I_{Ri}) + \left(\frac{\partial RC_{-i}(\mathbf{I_{R-i}})}{\partial \mathbf{I_{R-i}}} + \widetilde{\mu}_{h-i}[e^{j\alpha-i}]\right)\lambda_1 \\ &+ \lambda_2(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}) + (\widetilde{\mu}_h - \mathbf{K}^*[e^{j\theta}]^{-1}[Y_h]^{-1})\lambda_3 \end{aligned} \tag{3.47}$$

With associated first order conditions

$$\frac{\partial \mathcal{L}}{\partial V_{hi}} = (k_i^* - k_i) - \lambda_2 \begin{pmatrix} \widetilde{y}_{1i}e^{j\theta_i} \\ \widetilde{y}_{2i}e^{j\theta_i} \\ \vdots \\ \widetilde{y}_{ni}e^{j\theta_i} \end{pmatrix} = 0 \tag{3.48}$$

$$\text{Given that} \quad \lambda_2 = \frac{\partial \text{Utility of load } i}{\partial \mathbf{I_h}} \tag{3.49}$$

Equation 3.48 can be rewritten as

$$\begin{array}{c}\text{Marginal income to load } i \text{ due to}\\ \text{harmonic current injections}\end{array} = \begin{array}{c}\text{Marginal cost to load } i \text{ due to}\\ \text{harmonic current injections}\end{array}$$

$$\frac{\partial \mathcal{L}}{\partial I_{Ri}} = -\widetilde{\mu}_{hi} e^{j\alpha_i} - \frac{\partial RC_i(I_{Ri})}{\partial I_{Ri}} - \lambda_{2i} e^{j\alpha_i} = 0 \tag{3.50}$$

$$\Rightarrow \begin{array}{l}\text{Marginal income to load } i\\ \text{from harmonic reduction}\end{array} + \begin{array}{l}\text{Marginal utility to load } i\\ \text{from harmonic reduction}\end{array} = \begin{array}{l}\text{Marginal cost to load } i\\ \text{from harmonic reduction}\end{array}$$

For a large network it is generally considered that the marginal utility to load i resultant from the reduction of their own injections is small. This condition was assumed in Section 3.4, and equivalent to assuming that the load is responsible for a small fraction of the total injections into the network. Numerically this implies that λ_{2i}, is small with respect to $\|\mu_{hi}\|$.

$$\frac{\partial \mathcal{L}}{\partial k_i^*} = V_{hi} - (e^{j\theta_i}\widetilde{y}_{i1}^{-1}, e^{j\theta_i}\widetilde{y}_{i2}^{-1}, \ldots, e^{j\theta_i}\widetilde{y}_{in}^{-1})\boldsymbol{\lambda_3} = 0 \tag{3.51}$$

$$\Rightarrow \begin{array}{l}\text{Harmonic voltage}\\ \text{at busbar } i\end{array} = \begin{array}{l}\text{Resultant due to current}\\ \text{injections at all the busbars}\end{array}$$

Of particular interest is equation 3.48, which can be rewritten as

$$k_i^* = k_i + \boldsymbol{\lambda_2} \begin{pmatrix} \widetilde{y}_{1i} \\ \widetilde{y}_{2i} \\ \vdots \\ \widetilde{y}_{ni} \end{pmatrix} e^{j\theta_i} \neq k_i \quad \text{in general} \tag{3.52}$$

Equation 3.52 shows that in general there will be an incentive for a load to misstate their marginal costs due to harmonic distortion, in an effort to extract a profit out of other loads in the network. The degree to which this incentive will exist will depend the total share of voltage distortion at busbar i, the load at busbar i is responsible for. In the case where the load is small in the context of the given network ($\|\boldsymbol{\lambda_2}\| \approx \mathbf{0}$), the load has little incentive to misstate the harmonic valuation. This is as any extra return which can be squeezed out of the other members of the network are limited on account of the fact they will reduce their injections in the face of the increasing prices. On the other hand, a load which accounts for a significant proportion of the voltage distortion throughout the network, will have an incentive to misstate their harmonic valuations, in an attempt to claw back some of the payments they make to the other loads in the network.

This presents a potential pit fall in the use of marginal clean pricing of harmonic injections. Should there be a couple of large loads which are responsible for a large percentage of the total harmonic distortion, they will find it optimal to inflate their valuation of harmonic distortion ($u_i(V_{hi})$), in an attempt to minimise the harmonic payments made to other loads. This situation is analogous to the imperfect market situation as described by a Cournot equilibrium where in the presence of a competitive market with many firms, prices approach true marginal costs. But in the presence of a firm with considerable market share or market power, the resultant price will differ from true marginal costs [Varian1992].

In the presence of inflated marginal harmonic prices some loads will find it optimal to reduce their injections into the network when it is not efficient to do so, lowering the overall welfare of the network. But this loss of welfare is not fate accompli, as should there be many polluting loads each responsible for a small faction of the total distortion, there is no incentive for loads to misrepresent their valuation of distortion, and the marginal prices will closely represent the marginal valuation of inject harmonic current.

It should also be noted that the above analysis was of a long run nature. The model implies that loads instantly adjust their harmonic injections to the prices they see. In reality the process of reducing harmonic injections will not be instant, and in the case of larger loads is likely to require months rather than weeks. This lag between seeing the prices and responding means there may in fact be some incentive for all loads to inflate their valuation of harmonic distortion. As in the presence of a response lag a small load i ($\Rightarrow \|\boldsymbol{\lambda_2}\| \approx 0$), who receives no long term benefit from inflation of k_i, will be in a position to make excess profits from the other loads during the lag between seeing the prices and responding in the form of lower injections. Naturally the longer the response time from all loads the greater the incentive there will be for each to inflate their valuations of distortion. Hence there may be some need for oversight of a harmonic market to ensure that no individual with a short term horizon is in a position to prevent efficient market operation.

It is also possible to look at the incentives, which exist under marginal dirty pricing. Assuming that all the loads are small compared to the network, and the load at busbar i, is offered a marginal price $\widetilde{\mu}_{hi}$ for reduction in their harmonic injections, rational behaviour dictates that equation 3.42 be satisfied for each load. With the decision made with respect to harmonic reduction optimised according to equation 3.42, the other two things which effect each load's welfare are:

- Utility due to harmonic distortion seen at the local busbar $u_i(V_{hi}) = k_i V_{hi}$

- Payment to other loads for harmonic reduction $\sum_s^n k_i^* y_{is}^{-1} I_{Rs}$ (as described by equation 3.39)

Therefore under marginal dirty pricing the problem for the load at busbar i, can be expressed as:

$$\text{Maximise} \qquad V_{hi}k_i + k_i^* \mathbf{y_i}^{-1}\mathbf{I_R} \tag{3.53}$$

$$\text{Subject to} \qquad \widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h} = 0 \tag{3.54}$$

$$\frac{\partial RC(\mathbf{I_R})}{\partial \mathbf{I_R}} + \mathbf{K}^*[e^{j\theta}]^{-1}[Y_h]^{-1}[e^{j\alpha}] = 0 \tag{3.55}$$

Where $\widetilde{\mathbf{y}}_\mathbf{i}^{-1}$ = Row vector contianing row i of $[Y_h]^{-1}$
$\mathbf{y_i}^{-1}$ = Maginitude of $\widetilde{\mathbf{y}}_\mathbf{i}^{-1}$

This optimisation problem can be solved using the Lagrangian

$$\mathcal{L} = V_{hi}k_i + k_i^* \mathbf{y_i}^{-1}\mathbf{I_R} + \boldsymbol{\lambda_1}\left(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}\right) + \left(\frac{\partial RC(\mathbf{I_R})}{\partial \mathbf{I_R}} + \mathbf{K}^*[e^{j\theta}]^{-1}[Y_h]^{-1}[e^{j\alpha}]\right)\boldsymbol{\lambda_2} \tag{3.56}$$

The first order conditions for this problem are:

$$\frac{\partial \mathcal{L}}{\partial V_{hi}} = k_i - \lambda_1 \begin{pmatrix} \widetilde{y}_{1i} \\ \widetilde{y}_{2i} \\ \vdots \\ \widetilde{y}_{ni} \end{pmatrix} e^{j\theta_i} = 0 \tag{3.57}$$

$$\frac{\partial \mathcal{L}}{\partial k_i^*} = \mathbf{y_i}^{-1}\mathbf{I_R} + \mathbf{y_i}^{-1}\boldsymbol{\lambda}_2 = 0 \tag{3.58}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_R}} = k_i^* \mathbf{y_i}^{-1} - \lambda_1[e^{j\alpha}] = \mathbf{0} \tag{3.59}$$

In equation 3.59 it is assumed that the cost of harmonic reduction is linear function of the current magnitude.

$$\begin{aligned} \text{Given that} \begin{pmatrix} \widetilde{y}_{1i} \\ \widetilde{y}_{2i} \\ \vdots \\ \widetilde{y}_{ni} \end{pmatrix} e^{j\theta_i} &= \frac{\partial [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}}{\partial V_{hi}} = \frac{\partial(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R})}{\partial V_{hi}} \\ \text{Equation 3.57} \Rightarrow \lambda_1 &= k_i \frac{\partial V_{hi}}{\partial(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R})} \end{aligned} \tag{3.60}$$

Substituting equation 3.60 into 3.59 the result is

$$k_i^* \mathbf{y_i}^{-1} - k_i \frac{\partial V_{hi}}{\partial(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R})}[e^{j\alpha}] = \mathbf{0} \tag{3.61}$$

Then using the fact $\frac{\partial \widetilde{V}_{hi}}{\partial(\widetilde{\mathbf{I}}_\mathbf{h} - \widetilde{\mathbf{I}}_\mathbf{R})}[e^{j\alpha}]\frac{\partial V_{hi}}{\partial \widetilde{V}_{hi}} = \mathbf{y_i}^{-1}e^{j\beta^{-1}}[e^{j\alpha}]e^{-j(\beta^{-1}+\alpha)} = \mathbf{y_i}^{-1}$, equation 3.61 can be rewritten as

$$\begin{aligned} k_i^* \mathbf{y_i}^{-1} - k_i \mathbf{y_i}^{-1} &= \mathbf{0} \\ \Rightarrow k_i^* &= k_i \end{aligned} \tag{3.62}$$

This is the same result as was found for marginal clean pricing where $\|\boldsymbol{\lambda}_2\| = \mathbf{0}$. This is as is expected as the above result was developed under under the assumption equation 3.42 holds which as shown earlier is equivalent to saying $\|\boldsymbol{\lambda}_2\| = \mathbf{0}$.

3.7 CONCLUSION

This chapter demonstrated how using only information about the harmonic voltage at each busbar, and how the loads are effected by this voltage, it is possible to develop a set of prices for harmonic injections which accurately reflect the value of these injections to the rest of the network. The calculation of these prices is not trivial, as along with needing information as to the magnitude of the harmonic voltage, information is also required as to the phase of the voltage. This requires synchronous measurements throughout the network, which are generally unavailable. But with the continued development of harmonic state estimation [Arrillaga *et al.*2000] and cheaper metering technology, with time, measurement of these distortion voltages becomes more feasible.

The marginal harmonic prices are also of interest as they represent the first attempt to actually measure the value of injected harmonic currents. It is only with this value the network as a whole places on a given injection, is it possible to make optimal decisions, which maximise the utility of the network. As demonstrated when a load is presented with a marginal price, which is a true reflection of the cost of their actions, the load should behave in a manner which improves the welfare of every load in the network. But along with achieving an efficient outcome marginal pricing has the important property that it is inherently fair, as the prices only provide signals to which each load responds.

Given an allocation of harmonic property rights marginal pricing can only improve the welfare of each load. It should be noted though that the welfare of each load will be effected by the allocation of the harmonic property rights. The decision if each load has a right to a clean supply, or the right to inject what it wishes into the network will have a substantial effect on the utility of each network participant. This thesis in no way tries to suggest how the property rights should be allocated. But given the allocation of the property rights (which for every network is already done either explicitly or implicitly) the development and use of marginal prices can only improve the welfare of each load.

Marginal prices do have a weakness in that large polluting loads may well have an incentive to misstate their valuation of harmonic distortion, resulting in prices which no longer reflect the true cost to network. These distorted incentives are unlikely to lead to an optimal outcome. This potential problem though does not destroy all the value in marginal prices, as the distorted signals will lead the network towards an equilibrium, which is more efficient than before. That is while the incentive for some loads to misrepresent their harmonic valuations may result in full efficiency not being achieved, some improvement in overall network utility is still likely to occur. It should be noted though, with the distorted prices it is likely that the resultant harmonic payments will not be deemed as fair, as some loads will prosper at the expense of others. But as mentioned, in the situation where all loads are small compared with the network, there will be no incentive for the loads to misrepresent their distortion values and an optimal equilibrium should be achieved. In either event, some level of over sight to ensure some individuals are not attempting to prosper at the expense of others will ensure the marginal prices remain both efficient and fair.

Chapter 4

COMPLEX PRICES

4.1 INTRODUCTION

In Chapter 3 marginal prices for harmonic injections were developed under the assumption that all harmonic sources inject harmonic current into the network at a common angle. Clearly this restriction is unrealistic. While it is likely there will be certain distortion sources common to most of the loads in a network (computer power supplies, fluorescent lights), it is highly unlikely that the nonlinear loads at each busbar will be identical. This chapter investigates marginal pricing of harmonic injections where not all harmonic current is injected into the network with a common phase angle. Specifically considered are the incentives for some load z, which has the ability to inject harmonic current into the network at any possible phase.

Initially this chapter investigates the behaviour of the harmonic voltage magnitudes and angles as the angle of injection from load z (α_z), is varied. This behaviour is then used to formulate the marginal harmonic prices in this more general environment. An investigation of how these more general prices behave and how the payments amongst the loads vary, as α_z varies, is then carried out. Previously when all injections were at a common angle it was found that each load is conveniently charged a real amount for their injections. This is not the case when loads can inject into the network at any angle, as complex payments by loads result. An interpretation of these complex payments is developed along with a proof that only the real parts of these complex payments are important. Moreover using the test system an example of the marginal pricing in this more general envirnoment is constructed

4.2 VOLTAGES

It was shown in Section 3.2 that the voltages through out the network are given by the equation 4.1.

$$\begin{aligned}\widetilde{\mathrm{V}}_{\mathrm{h}} &= \begin{pmatrix} \sum_k^n \widetilde{y}_{1k}^{-1} e^{j\alpha_k} I_{hk} \\ \sum_k^n \widetilde{y}_{2k}^{-1} e^{j\alpha_k} I_{hk} \\ \vdots \\ \sum_k^n \widetilde{y}_{nk}^{-1} e^{j\alpha_k} I_{hk} \end{pmatrix} \\ &= \begin{pmatrix} \sum_k^n y_{1k}^{-1} e^{j(\beta^{-1}+\alpha_k)} I_{hk} \\ \sum_k^n y_{2k}^{-1} e^{j(\beta^{-1}+\alpha_k)} I_{hk} \\ \vdots \\ \sum_k^n y_{nk}^{-1} e^{j(\beta^{-1}+\alpha_k)} I_{hk} \end{pmatrix}\end{aligned} \tag{4.1}$$

Earlier when looking at these voltages α_k was fixed for all k. Here this constraint is relaxed, in that the load at busbar z will be allowed to inject current into the network at any angle. The result being:

$$\begin{aligned}\alpha_k &= \alpha \quad \forall\, k \neq z\\ \alpha_z &= \alpha_z\end{aligned}$$

Equation 4.1, is easily adjusted to account for this injection behaviour.

$$\widetilde{\mathbf{V}}_{\mathbf{h}} = \begin{pmatrix} \sum_{k\neq z}^{n} y_{1k}^{-1} e^{j(\beta^{-1}+\alpha)} I_{hk} + y_{1z}^{-1} e^{j(\beta^{-1}+\alpha_z)} I_{hz} \\ \sum_{k\neq z}^{n} y_{2k}^{-1} e^{j(\beta^{-1}+\alpha)} I_{hk} + y_{2z}^{-1} e^{j(\beta^{-1}+\alpha_z)} I_{hz} \\ \vdots \\ \sum_{k\neq z}^{n} y_{nk}^{-1} e^{j(\beta^{-1}+\alpha)} I_{hk} + y_{nz}^{-1} e^{j(\beta^{-1}+\alpha_z)} I_{hz} \end{pmatrix} \tag{4.2}$$

$$\text{Let} \quad \sum_{k\neq z}^{n} y_{ik}^{-1} I_{hk} = \gamma_i \tag{4.3}$$

$$y_{iz}^{-1} I_{hz} = \xi_i \tag{4.4}$$

Then equation 4.2 can be simplified to:

$$\widetilde{\mathbf{V}}_{\mathbf{h}} = \begin{pmatrix} \gamma_1 e^{j(\beta^{-1}+\alpha)} + \xi_1 e^{j(\beta^{-1}+\alpha_z)} \\ \gamma_2 e^{j(\beta^{-1}+\alpha)} + \xi_2 e^{j(\beta^{-1}+\alpha_z)} \\ \vdots \\ \gamma_n e^{j(\beta^{-1}+\alpha)} + \xi_n e^{j(\beta^{-1}+\alpha_z)} \end{pmatrix} \tag{4.5}$$

From equation 4.5, the voltage magnitude (Appendix C) and angle are easily derived.

$$\mathbf{V}_{\mathbf{h}} = \begin{pmatrix} [\gamma_1^2 + 2\gamma_1\xi_1 \cos(\alpha - \alpha_z) + \xi_1^2]^{\frac{1}{2}} \\ [\gamma_2^2 + 2\gamma_2\xi_2 \cos(\alpha - \alpha_z) + \xi_2^2]^{\frac{1}{2}} \\ \vdots \\ [\gamma_n^2 + 2\gamma_n\xi_n \cos(\alpha - \alpha_z) + \xi_n^2]^{\frac{1}{2}} \end{pmatrix} \tag{4.6}$$

$$tan(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\gamma_1 sin(\beta^{-1}+\alpha)+\xi_1 sin(\beta^{-1}+\alpha_z)}{\gamma_1 cos(\beta^{-1}+\alpha)+\xi_1 cos(\beta^{-1}+\alpha_z)} \\ \vdots \\ \frac{\gamma_n sin(\beta^{-1}+\alpha)+\xi_n sin(\beta^{-1}+\alpha_z)}{\gamma_n cos(\beta^{-1}+\alpha)+\xi_n cos(\beta^{-1}+\alpha_z)} \end{pmatrix} \tag{4.7}$$

4.3 MARGINAL PRICES

The fact that the harmonic injection angles are no longer constrained will not effect the expression from which the optimal marginal prices are calculated. That is the optimal marginal prices are still given by equation 3.8, i.e.

$$\widetilde{\boldsymbol{\mu}}_h = \frac{\partial U(\mathbf{V}_\mathbf{h})}{\partial \mathbf{V}_\mathbf{h}} [e^{j\boldsymbol{\theta}}]^{-1} [Y_h]^{-1}$$

Or in the case of linear utility functions equation 3.10, i.e.

$$\widetilde{\boldsymbol{\mu}}_h = \mathbf{K} [e^{j\boldsymbol{\theta}}]^{-1} [Y_h]^{-1}$$

The voltage angles from equation 4.7 can be substituted into the above expressions and prices calculated. The resulting equations though are complex and provide little insight into how the marginal prices behave. Looking at the expression for the voltage angle at busbar i.

$$\tan(\theta_i) = \frac{\frac{\gamma_i}{\xi_i} sin(\beta^{-1} + \alpha) + sin(\beta^{-1} + \alpha_z)}{\frac{\gamma_i}{\xi_i} cos(\beta^{-1} + \alpha) + cos(\beta^{-1} + \alpha_z)} \tag{4.8}$$

Equation 4.8, is of the form

$$tan(\theta_{hi}) = \frac{a + sin(\phi)}{b + cos(\phi)} \tag{4.9}$$

Where a and b are constants $\gg$ $sin(\phi)$ and $cos(\phi)$ respectively. Equation 4.9 has a simple graphical representation shown in figure 4.1.

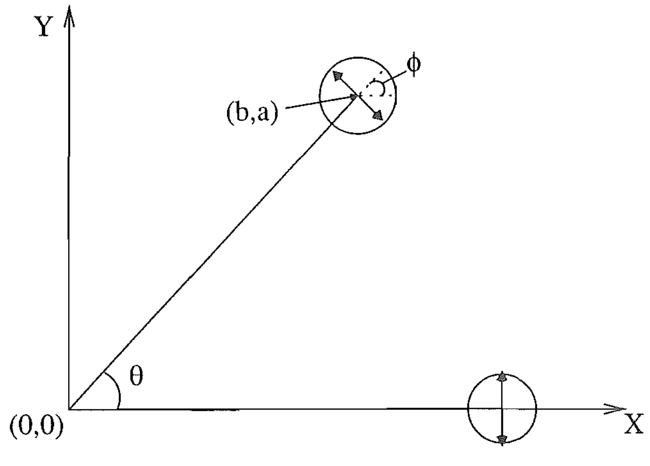


Figure 4.1 Geometric representation of the voltage angle

The expression for the voltage angle in 4.8 and 4.9 can be linearised using the Talyor series expansion of $\tan(\theta)$

$$\tan(\theta + \Delta\theta) \approx \tan(\theta) + \sec^2(\theta)\Delta\theta \tag{4.10}$$

Therefore in the case where $\theta = 0$

$$\tan(\Delta\theta) = \Delta\theta \tag{4.11}$$

Given that $a \gg sin(\phi)$, and $b \gg cos(\phi)$; we need only consider variation in the circle which is tangent to the line $(0,0) \rightarrow (b,a)$.

$$\begin{aligned} tan(\Delta\theta) &= \frac{y}{x} \\ &= \frac{sin(\phi)}{x} \\ &= \frac{sin(\phi)}{b} \quad \text{for } \theta = 0 \end{aligned} \tag{4.12}$$

Generalisation to any point on the plane

$$\Delta\theta = \frac{sin(\phi - \theta)}{\|(0,0) \to (b,a)\|} \tag{4.13}$$

Since $\tan(\Delta\theta) = \Delta\theta$ (4.11), applying this result to the voltage angle seen at some busbar i, gives:

$$\Delta\theta_i = \frac{sin(\alpha_z - \alpha)}{\frac{\gamma_i}{\xi_i}} \tag{4.14}$$

The result being the voltage angle at each busbar can be approximated by:

$$\theta_i = \beta^{-1} + \alpha + \frac{sin(\alpha_z - \alpha)}{\frac{\gamma_i}{\xi_i}} \tag{4.15}$$

Equation 4.15 is much simpler than 4.8, and can be used to derive manageable expressions for the resultant prices and payments made by the loads.

$$[e^{j\boldsymbol{\theta}}]^{-1} = \begin{pmatrix} e^{-j\left(\beta^{-1}+\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_1}{\xi_1}}\right)} & & \\ & \ddots & \\ & & e^{-j\left(\beta^{-1}+\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_n}{\xi_n}}\right)} \end{pmatrix} \tag{4.16}$$

In the case where each loads utility is a linear function of the votage magnitude at local busbar, the marginal price vector is given by:

$$\widetilde{\mu}_h = \mathbf{K} \begin{pmatrix} y_{11}^{-1} e^{-j\left(\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_1}{\xi_1}}\right)} & \cdots & y_{1n}^{-1} e^{-j\left(\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_1}{\xi_1}}\right)} \\ \vdots & \ddots & \vdots \\ y_{n1}^{-1} e^{-j\left(\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_n}{\xi_n}}\right)} & \cdots & y_{nn}^{-1} e^{-j\left(\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_n}{\xi_n}}\right)} \end{pmatrix} \tag{4.17}$$

As such the optimal marginal price for any busbar i is given by

$$\widetilde{\mu}_{hi} = \sum_j^n k_j y_{ji}^{-1} e^{-j\left(\alpha+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_j}{\xi_j}}\right)} \tag{4.18}$$

The amounts paid to each load for their injections are given by

$$\widetilde{\mu}_{hi}\widetilde{I}_{hi} = I_{hi} \sum_j^n k_j y_{ji}^{-1} e^{-j\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_j}{\xi_j}}} \qquad i \neq z \tag{4.19}$$

$$\widetilde{\mu}_{hz}\widetilde{I}_{hz} = I_{hz} \sum_j^n k_j y_{jz}^{-1} e^{-j\left(\alpha-\alpha_z+\frac{sin(\alpha_z-\alpha)}{\frac{\gamma_j}{\xi_j}}\right)} \tag{4.20}$$

4.4 RESULTING PAYMENTS

4.4.1 Optimality

Equations 4.19 and 4.20 are of interest, as unlike previously (equation 3.19), each of the loads will be charged a complex amount for their harmonic injections into the network. This posses problems from both a practical standpoint, and from theoretical standpoint. Just what is a complex payment and what does it represent? In solving this problem the first aspect to consider is that equation 3.11 indicates no matter what the angles of the injected harmonics the payments received from the distorting loads will equal those due to the effected loads, that is:

$$\begin{aligned} \widetilde{\mu}_h \widetilde{\mathbf{I}}_\mathbf{h} &= \mathbf{KV_h} \\ \text{Total payments} &= \text{Harmonic distortion costs} \end{aligned}$$

Implicit in this result is that the imaginary part of the payments due from all the loads sum to zero, as $\mathbf{KV_h}$ is a real number. This leads to the solution of ignoring the imaginary amount charged to each load on the basis that these imaginary payments sum to zero. While the correct amount of money will raised, before implementing harmonic charges this way, one needs to ensure ignoring the imaginary payments will still provide the correct incentives to each load, so that efficiency can be achieved. Will charging each load i the real part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$, encourage that load to act in an efficient manner?

To achieve efficiency, the pricing system must be such that for any load z, incentives are aligned with the system utility function. That is the payments made by the distorting load must be a minimum when the system voltage profile is minimised, and the payments made by the distorting load maximised when the system voltage profile is maximised.

The magnitude of voltage distortion at busbar i, is given in eqaution 4.6 as:

$$V_{hi} = [\gamma_i^2 + 2\gamma_i\xi_i \cos(\alpha - \alpha_z) + \xi_i^2]^{\frac{1}{2}} \tag{4.21}$$

The overall system harmonic utility ($\mathbf{KV_h}$) is maximised when the harmonic voltage distortion is minimised, and vice versa. From equation 4.6, it can be seen that the stationary points for the voltage magnitude at each busbar are coincident upon a common value of α_z. The stationary points of total network utility are given by:

$$\frac{\partial V_{hi}}{\partial \alpha_z} = \frac{\gamma_i\xi_i \sin(\alpha - \alpha_z)}{[\gamma_i^2 + 2\gamma_i\xi_i \cos(\alpha - \alpha_z) + \xi_i^2]^{\frac{1}{2}}} = 0 \tag{4.22}$$

This suggests the voltage magnitude at each busbar the stationary points at $\alpha_z = \alpha \pm n\pi$

$$\begin{aligned} \text{For } \alpha_z &= \alpha \pm 2n\pi \\ V_{hi} &= \gamma_i + \xi_i \to \max \\[1em] \text{For } \alpha_z &= \alpha \pm (2n-1)\pi \\ V_{hi} &= \gamma_i - \xi_i \to \min \\ & \quad \text{(where n is an interger)} \end{aligned}$$

Having established the optimal values of α_z from the network's point of view, the next step is to compare this with the incentives provided to the load at busbar z. The real and imaginary parts of the payments due from a load at busbar z ($\widetilde{\mu}_{hz}\widetilde{I}_{hz}$) are shown in equation 4.23 and 4.24.

$$Re\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} = I_{hz}\sum_{j}^{n} k_j y_{jz}^{-1} \cos\left(\alpha - \alpha_z + \frac{\xi_j}{\gamma_j}\sin(\alpha_z - \alpha)\right) \tag{4.23}$$

$$Im\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} = -I_{hz}\sum_{j}^{n} k_j y_{jz}^{-1} \sin\left(\alpha - \alpha_z + \frac{\xi_j}{\gamma_j}\sin(\alpha_z - \alpha)\right) \tag{4.24}$$

Given that it is proposed to only charge loads the real part of $\widetilde{\mu}_{hz}\widetilde{I}_{hz}$, to find the likely behaviour of the loads in this situation the stationary points of equation 4.23 have to be found.

$$\frac{\partial Re\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\}}{\partial \alpha_z} = -I_{hz}\sum_{j}^{n} k_j y_{jz}^{-1}\left(-1 + \frac{\xi_j}{\gamma_j}\cos(\alpha_z - \alpha)\right)\sin\left(\alpha - \alpha_z + \frac{\xi_j}{\gamma_j}\sin(\alpha_z - \alpha)\right) = 0 \tag{4.25}$$

From 4.25 it can seen that stationary points exist at $\alpha_z = \alpha \pm n\pi$, where

$$\begin{aligned} \text{For } \alpha_z &= \alpha \pm 2n\pi \\ Re\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} &= I_{hz}\textstyle\sum_j^n k_j y_{jz}^{-1} \rightarrow \min \\ \\ \text{For } \alpha_z &= \alpha \pm (2n-1)\pi \\ Re\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} &= -I_{hz}\textstyle\sum_j^n k_j y_{jz}^{-1} \rightarrow \max \\ & \quad (as\ k_j < 0 \quad \forall j) \end{aligned}$$

This demonstrates that by charging each load z the real part of $\widetilde{\mu}_{hz}\widetilde{I}_{hz}$ encourages behaviour that will minimise the harmonic voltages around the system. This suggests charging each load z, only the real part of equation 4.20 is a viable solution to the problem of complex payments. Charging each load only the real part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$ is easily implimented and such a structure will collect the correct amount of harmonic payments, while still encouraging efficient behaviour.

4.4.2 Payment Characteristics

Each load i only being charged the real part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$, produces some interesting results with respect to how the payments from each load change, as angle of injected harmonics from load z (α_z) changes.

First looking at a load $i \neq z$, which injects at the common angle α, the real and imaginary parts of what they are charged is given by

$$Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} = I_{hi}\sum_{j}^{n} k_j y_{ji}^{-1} cos\left(\frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha)\right) \tag{4.26}$$

$$Im\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} = -I_{hi}\sum_{j}^{n} k_j y_{ji}^{-1} sin\left(\frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha)\right) \tag{4.27}$$

To easily get insight into how these terms behave as α_z is varied equations 4.26 and 4.27 are linearised to give:

$$\text{As } \frac{\xi_j}{\gamma_j} \longrightarrow 0 \Rightarrow \quad \begin{array}{l} cos\left(\frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha)\right) \longrightarrow 1 \\ sin\left(\frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha)\right) \longrightarrow \frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha) \end{array}$$

Equations 4.26 and 4.27 can be simplified to yield:

$$Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} \approx I_{hi}\sum_{j}^{n} k_j y_{ji}^{-1} \tag{4.28}$$

$$Im\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} \approx -I_{hi}\sum_{j}^{n} k_j y_{ji}^{-1}\frac{\xi_j}{\gamma_j} sin(\alpha_z - \alpha) \tag{4.29}$$

Equations 4.28 and 4.29 suggest an interesting result. By only charging each load $i \neq z$ the real part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$ the amount paid is independent of α_z. That is, in a large network where all the loads are small relative to the network, the amount of money due from the other loads is unaffected. This intuitively seems fair, in that no small load should be able to enter a network and substantially affect the payments due from the others loads. Of course the fact that there is a new load in the network injecting harmonics at a different angle will change the harmonic voltage profile throughout the system, and this is reflected in the complex part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$. It should be noted though, should the load z, be responsible for a significant proportion of the harmonic distortion, then the linearisation becomes invalid and the load $i \neq z$, will have payments affected by α_z. The question of what is a significant prportion of the distortion is dealt with in Section 4.5.

To see how the payments due from a load z (which injects at a different angle to the rest of the system) behave, equations 4.23 and 4.24 are simplified using the fact $\frac{\xi_j}{\gamma_j} \longrightarrow 0$. This gives

$$Re\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} \approx I_{hz}\sum_{j}^{n} k_j y_{jz}^{-1} cos(\alpha - \alpha_z) \tag{4.30}$$

$$Im\{\widetilde{\mu}_{hz}\widetilde{I}_{hz}\} \approx -I_{hz}\sum_{j}^{n} k_j y_{jz}^{-1} sin(\alpha - \alpha_z) \tag{4.31}$$

Naturally the payments made by load z will be dependent on the angle of the injections. As demonstrated before the payments due are such, that load z is encouraged to act in an optimal way ie. set $\alpha_z = \alpha \pm \pi$ if possible.

The above results may seem a bit contrived in the sense that they were developed based on the assumption that all loads inject at a common angle, except one, but the result is more general then the under lying assumptions may suggest. The result shown in Section 4.4.1, that any small load has incentives aligned with that of the whole network holds irrelevant of what angles the other harmonic injections are made. Similarly the result in equation 4.28 that payments by any load with constant injection angle will be largely unaffected by the injections of some load z, whose angle may change, is true for any large network where no single large loads have the ability to substantially effect the harmonic voltage profile.

This leaves the question of what does the imaginary part of the complex payments represent? Initially when considering the case where all loads were injected at a common angle it was stated the marginal price at a busbar represented, the marginal value to the whole network of injections at that busbar is $\widetilde{\mu}_h = \dfrac{\partial \text{Network Utility}}{\partial \widetilde{\mathbf{I}}_\mathbf{h}}$. Hence the total payments made by a load represent the cost those injections have to all members of the network. This result was a product of the assumption, that all the injections made by a load are in phase with the prevailing voltage (actually considering the network looks overwhelmingly inductive the current injections are about ninety degrees out of phase with the voltage). Another interpretation of the payments is that they were complex, with a imaginary component equal to zero. Considering that all the injections were in phase with the prevailing voltage, the conclusion then follows that the real part of the complex payments made is the value to the network of injections, which are in phase with the voltage. As such the imaginary part of the payments is related to the quadrature component of the injections. As this quadrature component does not effect the prevailing voltage magnitude (at the margin) its value is zero (hence why ignoring it, still encourages an efficient result). As such imaginary part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$ is the value of the quadrature current had it been in phase with the prevailing voltage.

4.5 VALIDITY CHECK

All the previous conclusions made about the behaviour of the harmonic load payments is implicitly based on the assumption that the linearisations preformed are valid. Linearisations of the voltage angle θ_{hi} and of the real and imaginary parts of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$, were used to simplify these expressions so that insights into how the marginal harmonic prices behaved could be gained. Naturally these insights are only as good as the assumptions used to develop them.

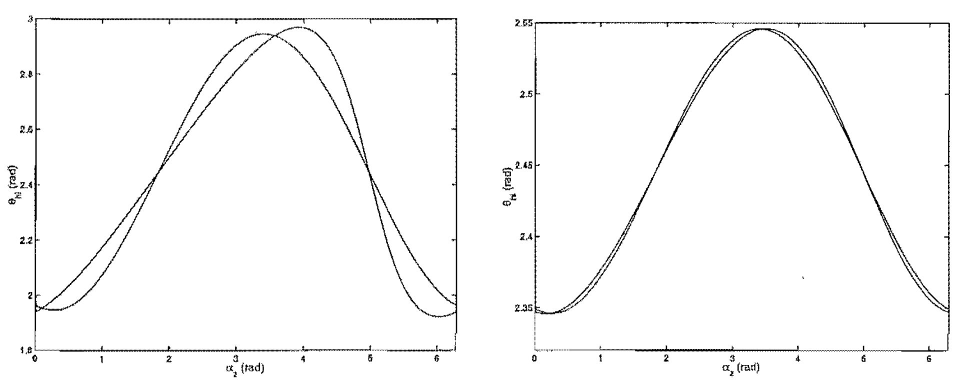
In Section 4.3 it was shown so long as $\frac{\gamma_i}{\xi_j} \gg 1$, then at any busbar i

$$tan(\theta_{hi}) = \frac{\frac{\gamma_i}{\xi_i} sin(\beta^{-1}+\alpha) + sin(\beta^{-1}+\alpha_z)}{\frac{\gamma_i}{\xi_i} cos(\beta^{-1}+\alpha) + cos(\beta^{-1}+\alpha_z)}$$

Can be approximated by the the expression

$$\theta_{hi} = \beta^{-1} + \alpha + \frac{sin(\alpha_z - \alpha)}{\frac{\gamma_i}{\xi_i}}$$

This should be the case for a large network with a great number of harmonic injecting loads. How good the approximation is demonstrated in Fig. 4.2, where the actual and aproximated voltage angle are shown as α_z is varied, for two different ratios of $\frac{\gamma_i}{\xi_i}$



(a) $\gamma_i/\xi_i = 2$ (b) $\gamma_i/\xi_i = 10$

Figure 4.2 Comparision of the angles given by equations 4.8 and 4.15

Fig. 4.2 shows that the linearisation does produce a good approximation of the voltage angle so long as there are a significant number of loads in the system. Even in the case where there are only three loads in the network ($\gamma_i/\xi_i \approx 2$), the linearisation produces results which are relatively close to the actual.

It was shown in Section 4.4 that the payments made to the harmonic sources are approximately given by

$$Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} = \begin{array}{ll} I_{hi}\sum_j^n k_j y_{ji}^{-1} & \text{if } i \neq z \\ I_{hi}\sum_j^n k_j y_{ji}^{-1}\cos(\alpha - \alpha_z) & \text{if } i = z \end{array} \tag{4.32}$$

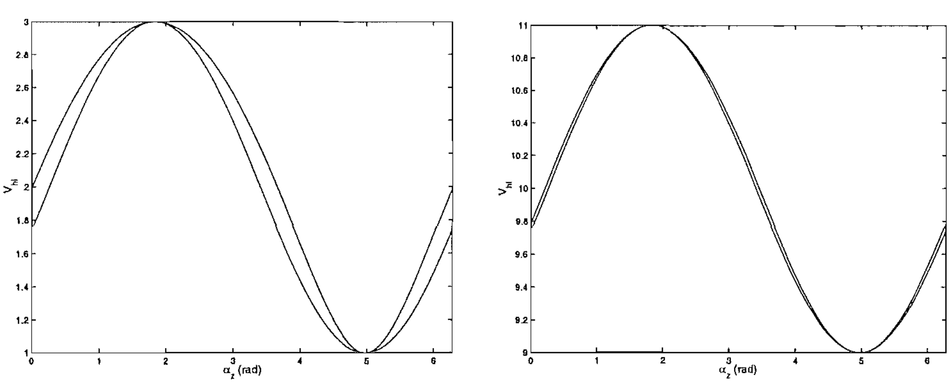
Equation 4.32 was used to make some generalisations about how the payments from each load varied as α_z varied. To check that these generalisations are valid must show that the total payments due as described by equation 4.32, are equal to the total due as described by $\mathbf{KV_h}$, i.e.

$$\sum_j^n k_j\gamma_j + \sum_j^n k_j\xi_j\cos(\alpha - \alpha_z) \approx \sum_j^n k_j[\gamma_j^2 + 2\gamma_j\xi_j\cos(\alpha - \alpha_z) + \xi_j^2]^{\frac{1}{2}} \tag{4.33}$$

This can be simplified to showing that

$$\gamma_j + \xi_j cos(\alpha - \alpha_z) \approx [\gamma_j^2 + 2\gamma_j\xi_j cos(\alpha - \alpha_z) + \xi_j^2]^{\frac{1}{2}} \tag{4.34}$$

Both terms in equation 4.34 represent the voltage magnitude at some busbar j. The term on the left of equation 4.34 being the simplifed version resulting from the linearisation of the voltage angle and assumptions about the ratio γ_j/ξ_j. Graphs of both the actual and simplied voltage magnitude terms are shown in Fig. 4.3 for different relative values of γ_j and ξ_j. As expected the approximation is better the larger the ratio between the two. But even for small ratios the approximation looks to be valid.



(a) $\gamma_i/\xi_i = 2$ (b) $\gamma_i/\xi_i = 10$

Figure 4.3 Comparision of the magintudes given in equation 4.34

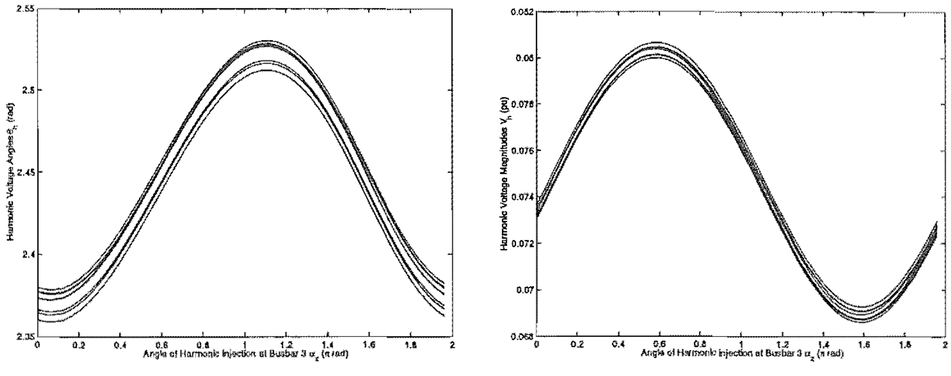
Given that it has been shown that the simplifed version of $\sum_i^n Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} \approx \mathbf{KV_h}$, the conclusions drawn in section 4.4 can be relied upon to portray their true behaviour.

4.6 EXAMPLE

Using the test system detailed in Appendix A, this section investigates the behaviour of the harmonic marginal prices and the payments amongst the loads, as the injection angle of the load at busbar three (α_3) is varied from $0 \rightarrow 2\pi$. The phase of harmonic injections made by all other loads is kept constant.

The harmonic voltage magnitude and angle, shown in Figure 4.4, are approximately equal at each busbar and have a sinusoidal dependence on α_3. This sinusoidal dependence and 90°

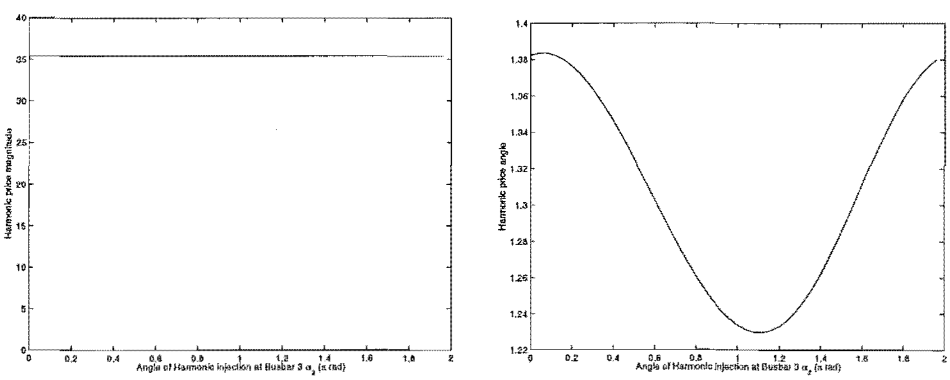
phase shift between the harmonic voltage magnitude and angle is exactly that suggested by the linearised equations 4.15 and 4.34.



(a) θ_{hi} variation (b) V_{hi} variation

Figure 4.4 Busbar voltages as α_z is varied

Figures 4.5 and 4.6 show how the complex price behaves in response to variation in α_3. Again there is a very close correlation with that suggested by the linearised price in equation 4.18. In this example the absolute value of the resultant harmonic prices is independent of α_3. Only the phase of the marginal prices show a sinusoidal dependence on α_3. Had the utility of each load not been specified as a linear function of the voltage distortion seen at their busbar, this would not have been the case.



(a) Harmonic price magnitude μ_{hi} (b) Harmonic price angle $\angle\widetilde{\mu}_{hi}$

Figure 4.5 Harmonic Price magnitude and angle as α_z is varied

As each load was given a linear harmonic utility function ($u_i(V_{hi}) = k_i V_{hi}$), the sinusoidal variation in the harmonic voltage magnitude will also be seen in the compensation payments made to each load (Figure 4.7).

Given that the compensation payments made to each load as shown in Figure 4.7 have a sinusoidal dependence on α_3, the total payments made by all loads will have a similar dependency. This is shown in Figure 4.8, along with the total payments made by all the loads. As shown previously the summation of the real part of all the complex charges $\sum Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}$, is equal to the total compensation payments due $\mathbf{KV_h}$.

In Section 4.4.2, it was suggested that in the case when all loads are small compared to the

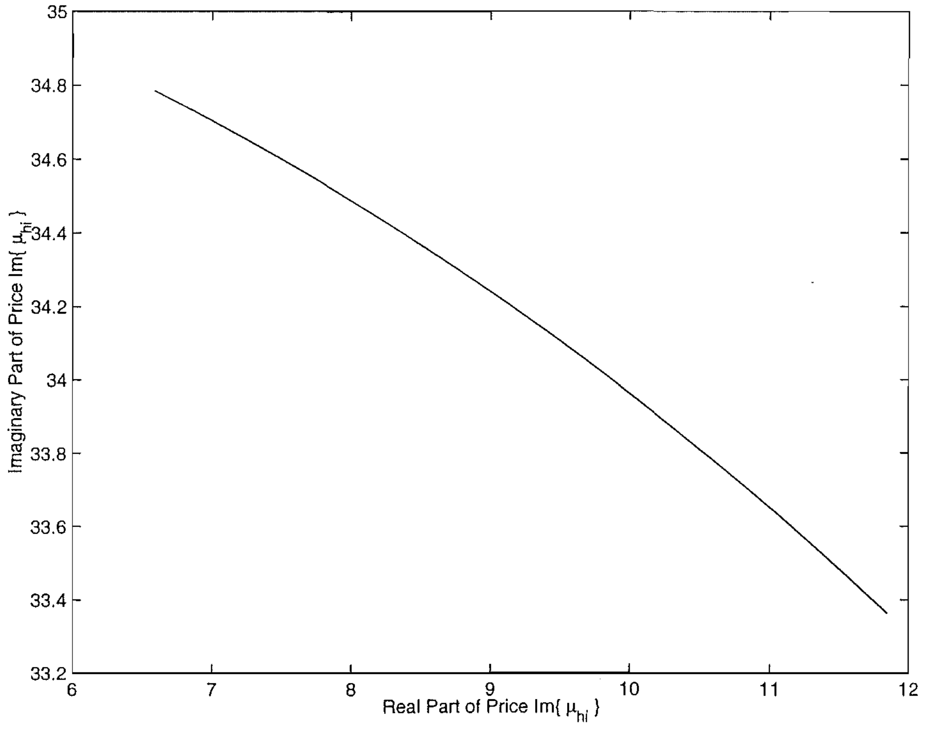


Figure 4.6 Complex price $\widetilde{\mu}_{hi}$ as α_z is varied

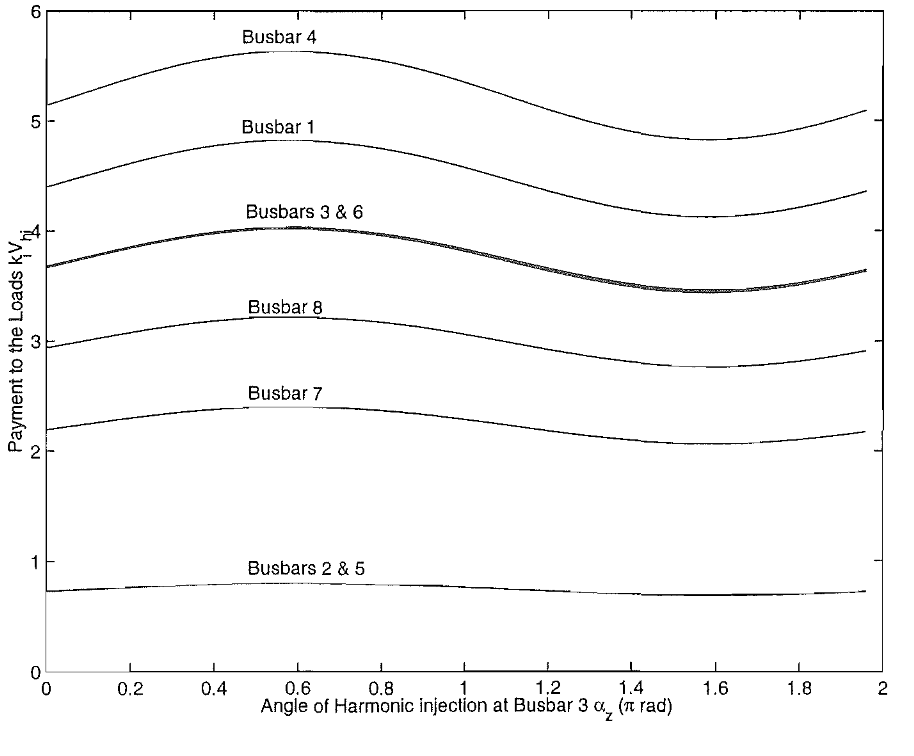
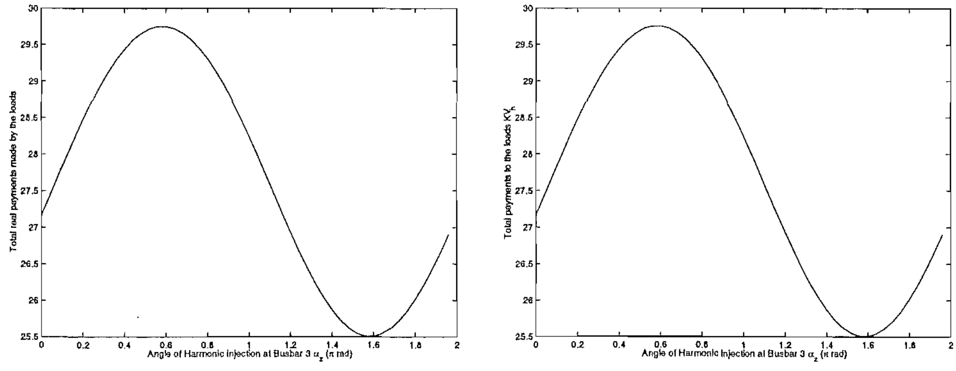


Figure 4.7 Payments made to each load as α_z is varied



(a) Total payments made by the loads $\sum Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}$

(b) Total payments made to the loads $\mathbf{KV_h}$

Figure 4.8 Total payments made to and by the loads as α_z is varied

network (or at least not responsible for more than a third of distortion), the payments from each load will be independent of the actions of the other loads in the network (at the margin). This is clearly shown in Figure 4.9, where the payments due from all the loads $i \neq 3$, $(Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}\forall\, i \neq 3)$ are stationary, while the load at busbar three, faces payments which directly reflect the value of his injections to the rest of the network (as shown in Figure 4.8). This property that the value of the different harmonic injection angles is clearly signalled to each load, is required for efficiency to be achieved. This allows loads to make efficient decisions not just with respect to the magnitude of their injections but also as to the phase of their injections. Admittedly there are very few situations where a load has any sort of control over the angle of harmonic injections, but in planning a new nonlinear load these pricing signals will improve the welfare of the whole network.

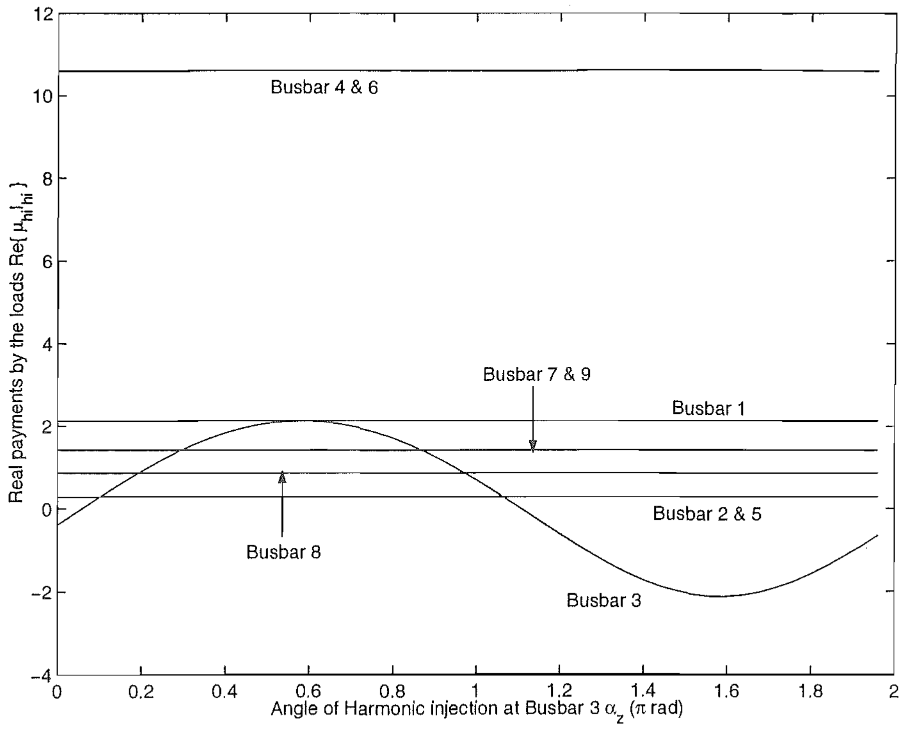


Figure 4.9 Real payments made by each load as α_z is varied

Figures 4.10 and 4.11 show the complex and absolute payments due from each load. Of note is that the absolute payment due from each load is a fixed amount equal to that which would be paid if all the loads inject at a common angle. This provides further evidence that making the value of the injection angle transparent can only improve overall welfare. The example suggests the linearised model developed accurately describes the system, and the conclusions drawn are valid.

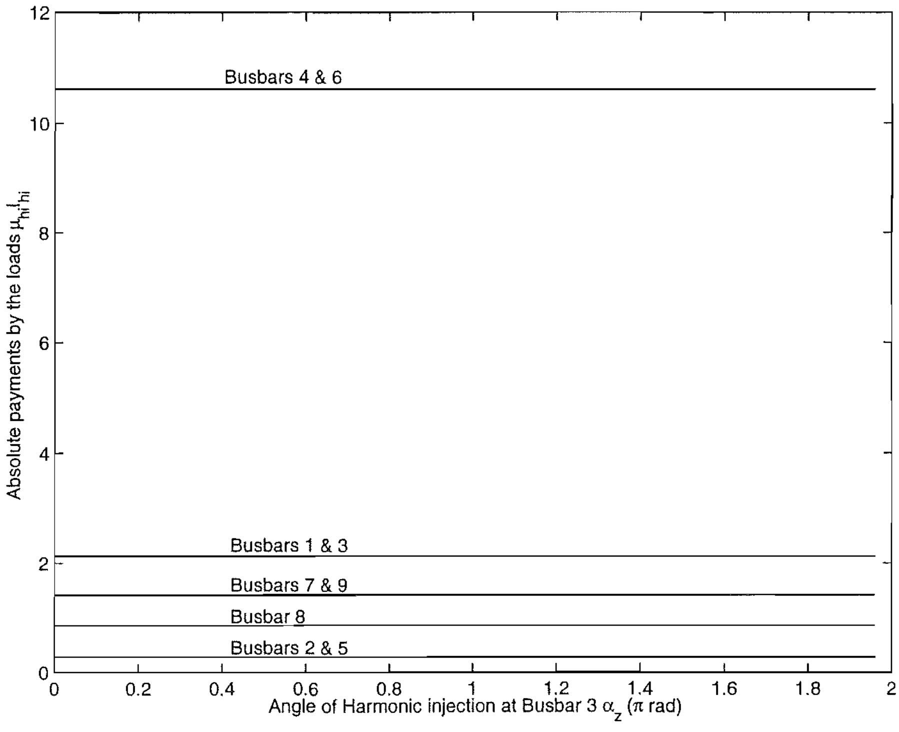


Figure 4.10 Absolute amount charged to each load ($\widetilde{\mu}_{hi}\widetilde{I}_{hi}$) as α_z is varied

4.7 MARGINAL DIRTY PRICING REVISTED

The previous example was carried out on the basis that each load has the right to a clean voltage supply. As discussed in Section 3.5, it is also possible to allocate the harmonic property rights so loads have the right to inject what they wish into the network (marginal dirty pricing). Here the payments made by each load under marginal dirty pricing are derived.

Consider the injections into the network by a load at busbar s. The injections made by the load at busbar s, have a value to the rest of the network given by equation 4.35

$$\begin{aligned}\text{Value to network of injections by load } s &= Re\{\widetilde{\mu}_{hs}\widetilde{I}_{hs}\} \\ &= Re\{\left(\sum_i k_i \widetilde{y}_{is}^{-1} e^{-j\theta_i}\right)\widetilde{I}_{hs}\}\end{aligned} \tag{4.35}$$

From this it naturally follows:

$$\text{Payment by load } z = Re\{k_z \widetilde{y}_{zs}^{-1} e^{-j\theta_z}\widetilde{I}_{hs}\} \tag{4.36}$$

In the case where the network is strong, so the voltage will be approximately equal through

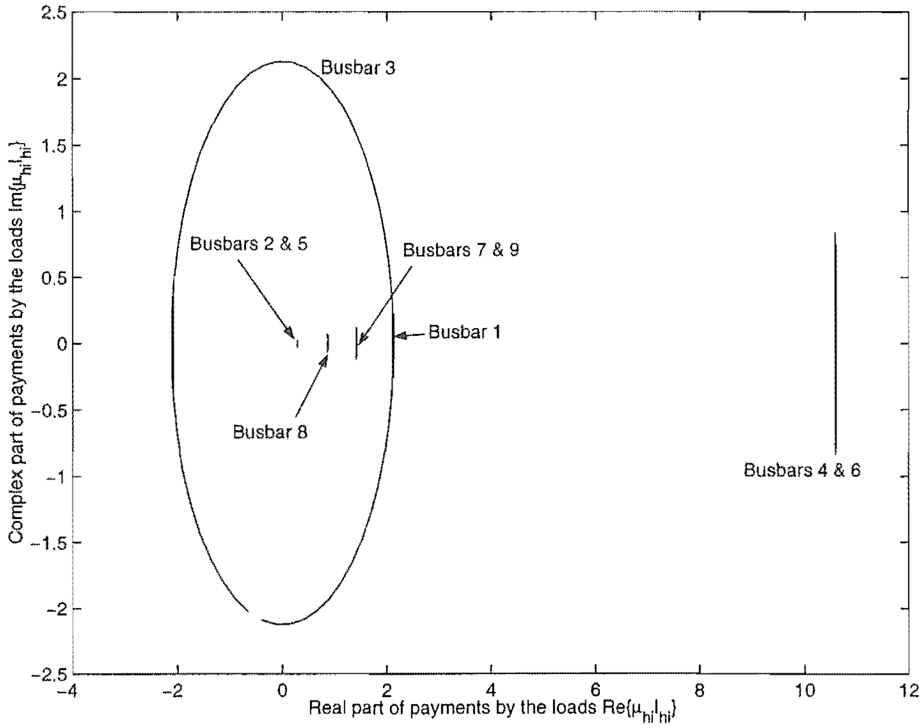


Figure 4.11 Complex amount charged to each load as α_z is varied

out, equation 4.36 can be simplified to:

$$\text{Payment by load } z = \frac{k_z y_{zs}^{-1}}{\sum_i k_i y_{is}^{-1}} Re\{\widetilde{\mu}_{hs}\widetilde{I}_{hs}\} \tag{4.37}$$

Equation 4.37, is very similar to equation 3.38, representing the general solution of the special case detailed in equation 4.37.

4.8 CONCLUSION

In the general case each load will be charged a complex amount for their harmonic injections. In the context of how we think about money and value this may seem nonsensical, but that is as as we generally live in a one dimensional economy, or we structure it to look one dimensional. Harmonic injections into a network on the other hand is a two dimensional quantity. Current has magnitude and phase, and hence any attempt to price this commodity and the resulting payments must in turn have two dimensions.

In the previous chapter, specifying that all injections are made at a common angle, is equivalent to constraining the problem to a one dimensional form (single line across a plane of possible inputs), the result being single dimensional payments. Generally when an economic problem has multiple dimensions the solution space is manipulated to reduce the problem back down to a single dimension. Often this is done by artificially creating numerous different commodities from the single commodity to allow for the extra dimension in the problem. Where each created commodity is a restricted case of the more general commodity (ie. different lines on the plane). For example looking at petrol, here there is a multi-dimensional problem, in that both the volume and octane rating of petrol affect its value. To allow petrol to easily fit into a single dimensional currency system, two different categories are created: low octane and high octane. Within these categories the two different types of petrol can be prices linearly. But if petrol is considered as a single commodity, the result is that all petrol must have a common price and splitting petrol in two sub products, is simply looking at two different constrained input spaces. In a similar way the pricing of harmonic current can be reduced to a single dimension problem by only considering the in phase component of all current injections. The point being that because a complex amount is charged to each load, does not necessarily mean things do not reconcile. This occurs all the time (in fact with any commodity where quality is not consistent), except instead of expressing our prices and payments in multi dimensional form, we create seperate discrete commodities to remove the imaginary part. As an aside it is of interest to consider the consequences of two dimensional or complex utility functions would be complex prices and payments for everything.

Having established that for a two dimensional commodity, one should expect two dimensional prices and payments, the next step is to understand what the real and imaginary amounts charged to each load represent. As was shown, the real part of the load represents the value of the in phase current injected by the load, while the imaginary part attaches a value to the quadrature component of the injection. That is the imaginary part of $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$, charged to each load i, indicates the value of the quadrature component, had it been in phase with the prevailing voltage. But as the quadrature component at the margin has no effect on the prevailing voltage, and hence the utility of the network, this value is specified as the imaginary component. The fact that the imaginary part has no value allows us to discard this part, and only charge each load the real part of the complex amount. Not only does this raise the correct amount of money, but provides the correct incentives to each load so efficiency can be achieved.

Also demonstrated was that so long as all loads are relatively small with respect to the network, the amount paid by each load at the margin is independent of the actions of all the other loads. This is very convenient as most loads would probably be unhappy at the thought their harmonic charges are beyond their control, and sensitive to the actions of others. But at a macro level the value of any injection is directly dependent on the actions of others in the network, and the marginal prices reflect this allowing efficiency to be achieved. In other words, the amount charged to a static load should will not vary rapidly, period to period, due to the actions of any other single load, instead only moving slowly due to a change in the state of the whole network.

Chapter 5

ACTIVE FILTER INCLUSION

5.1 INTRODUCTION

The primary motivation behind the development of marginal prices for harmonic injections, is to encourage an efficient allocation of resources towards harmonic mitigation. This allocation of resources can take two forms of action:

1. Efforts by individual loads to reduce harmonic current injected into the system
2. Installation of harmonic mitigation equipment (filters) in the network

Efficient allocation of resources requires an efficient split of resources between the two actions, and that each action be carried out optimally. To this point only the first action has been considered. The installation of a filter will dramatically reduce the voltage distortion throughout the network. Therefore the ability of marginal pricing to encourage efficient behaviour with respect to the allocation of filter resources is critical if marginal pricing is to prove efficient.

As well as efficiency, fairness is also important. It is important that marginal pricing distributes the costs of any installed filter throughout the network in a manner deemed fair.

In this chapter active filters in a marginal harmonic pricing environment are investigated. For the purposes of this chapter, active filters are modelled as controllable current sources (Figure 5.1). This restriction is relaxed in Chapter 7.

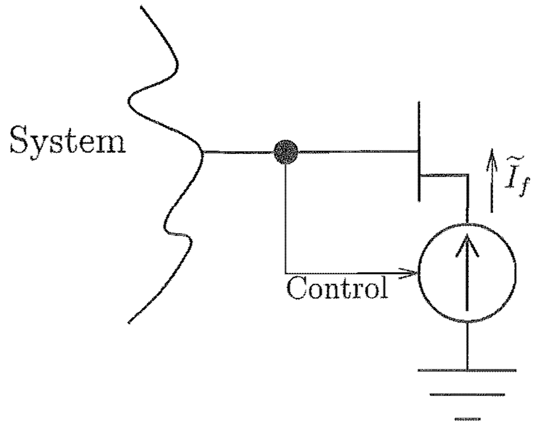


Figure 5.1 Active harmonic filter model

Where active filter injection at busbar $i = \widetilde{I}_{fi} = I_{fi}e^{j\psi_i}$

In this chapter the assumed cost structure for any installed active filter is specified. Given this structure, the optimal filter resource allocation is identified and compared with the incentives that exist for any potential filter owners. Here both single period and multi-period optimization situations are considered. It is found the provision of filter services violates some our previous assumptions, as the filter has such a large effect on the harmonic voltages throughout the network. The consequences of this are considered. As discussed in Chapter 2, there have been a number of methods put forward for allocating filter costs. An example is constructed and a comparison is made between the cost allocation resulting from marginal pricing (marginal clean), compared with the Toll Road method.

5.2 OPTIMAL ACTIVE FILTER

In the development of the conditions that describe the optimal allocation of active filter resources, the possibility is left open that there could be an active filter built at any, and all busbars. For various reasons, all busbars in a network may not be feasible sites for a filter. Moreover in the presence of diminishing marginal capital costs (should they exist), it is likely that one large filter is optimal for a strong network (as opposed to a number of smaller filters). If required the problem is easily adjusted so filters can be placed at limited number of busbars. But for the purposes of this work the solution is kept as general as possible. It should be noted if the cost structure of an active filter make a single large filter preferable to many small filters, this should become evident from the general solution.

5.2.1 Pricing Time Period

To this point the decision making process has always been described as a single period optimisation problem, without any reference to the length of this time period (referred to hence forth as the billing period). In one sense it would be is convenient to think of a billing period as one year, though it can be any length of time, and there may be good reason for not choosing a year. Given a billing period the next consideration is how often the prices are updated (pricing period). The behaviour described in Chapter 3 was based on the assumption that there is only a single price during the billing period, ie. the pricing period and billing period are equal. From the point of view of describing the behaviour of the network participants this assumption is very natural. While the billing period can differ to the pricing period, this separation is very artificial, and for the purposes of describing individual behaviour, once one has been established so has the other. As such the question arises what is a reasonable pricing period?

As the types of load in a network change slowly, the value a network places on voltage distortion will also only change slowly. That is, the network is likely to value some given level of harmonic distortion identically in six months or a year as it does today. This suggests that an appropriate pricing period could be a year, as the aggregate network preferences are fairly static over the period of a year.

There is a trade off between the length of the pricing period and the level of efficiency achieved. This is because the amount charged to a load during a pricing period must be based on their mean current injections, and the mean harmonic state of the network over the period. It was previously demonstrated that the value the network places on injections is given by equation 3.8, ie.

$$\widetilde{\mu}_h = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}[e^{j\boldsymbol{\theta}}]^{-1}[Y_h]^{-1}$$

The above equation states that in general the value of any injection will be dependent on the harmonic state of the network, that is the voltage magnitude and the voltage angle. For the marginal prices to encourage efficient behaviour they must accurately reflect the true value of the injections. It is highly unlikely that the harmonic state of the network will be constant for a period of a year, using the mean harmonic state over a year as the basis for an annual price, is unlikely to encourage efficient behaviour. To encourage efficient behaviour, there is a need for a pricing period closer to an hour. Given network preferences that are reasonably constant a pricing period of an hour is quite feasible (or at least as feasible as a pricing period of a year, as the same amount of information on the harmonic state of the network is required for both). As an efficient allocation of resources is one of the primary purposes for using marginal pricing, clearly it is desirable to use a short pricing period such as an hour.

Having established that for marginal pricing of harmonic injections to be efficient, the pricing/billing period must be in the order of an hour, what implications does this have for the load behaviour described before in Chapter 3. Basically it has little influence on the previous results, due to the way te problem was structured. Previously it was assumed that loads could take some measure to reduce their injections into the network at some cost (described by $RC_i(I_{Ri})$ for load i). There was no suggestion that these measures were permanent. As such all the previous results in Chapter 3, which resulted from a single period optimisation problem still hold in a dynamic world (under the no permanence assumption).

The situation changes slightly when filters or other multiple period measures are introduced into the problem. Here a decision has to be made as to the amount of capital to be invested in an asset which will deliver returns over multiple periods. This added dimension, and how it effects the decision making process for a network, is detailed in the next section. It should be noted that in this work the pricing period is purposely left unspecified. The ideal pricing period will almost certainly depend on the circumstance and the technology available. Any filter installed in a network will have an operating life in excess of a single pricing period.

5.2.2 Network Optimum

Despite having just stated that the problem of finding the optimal filter is a dynamic optimisation problem, for simplicity this section starts off by looking at the single period case. In finding the optimal active filter(s), the cost associated with such filters must be specified. Here it is assumed that for each time period the active filter at a busbar i, has a operating cost associated with the magnitude of the current injected into the network.

$$\text{Operating Cost of Filter at Busbar } i = FV_i(I_{fi}) \tag{5.1}$$

In all likelihood the operating cost will be essentially zero, but it is included for completeness. The major cost associated with an active filter is the capital cost associated with its construction, and on going maintenance and inventory costs. These costs are assumed to be dependent on the rating/capacity of the filter. That is, the capital costs are assumed to be a function of the maximum possible filter current I_{fimax}. Given an annual cost of capital and service life of the filter, this lump sum capital cost can easily be converted to a fixed term annuity, based on a compounding period equal to the pricing period.

$$\text{Cost of Filter Capital For Filter at Busbar } i = FF_i(I_{fimax}) \tag{5.2}$$

$$\therefore \text{Total Cost For Filter at Busbar } i = FV_i(I_{fi}) + FF_i(I_{fimax}) \tag{5.3}$$

Using the cost functions for the filter at each busbar (should they exist), network filter cost functions are easily specified.

$$\text{Total Operating Cost of Active Filter Injections} = FV(\mathbf{I_f}) = \sum_i^n FV_i(I_{fi}) \tag{5.4}$$

$$\text{Total Capital Cost of Active Filter Injections} = FF(\mathbf{I_{fmax}}) = \sum_i^n FF_i(I_{fimax}) \tag{5.5}$$

The network utility maximisation problem in the case where active filters are allowed to be installed at any busbar is described by:

$$\text{Maximise} \quad U(\mathbf{V_h}) - FV(\mathbf{I_f}) - FF(\mathbf{I_{fmax}}) \tag{5.6}$$

$$\text{Subject to} \quad \widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{f} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h} = \mathbf{0} \tag{5.7}$$

$$\mathbf{I_f} \leq \mathbf{I_{fmax}} \tag{5.8}$$

This problem can be solved using the Lagrangian given in equation 5.9

$$\mathcal{L} = U(\mathbf{V_h}) - FV(\mathbf{I_f}) - FF(\mathbf{I_{fmax}}) + \widetilde{\boldsymbol{\mu}}_h(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{f} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}) + \boldsymbol{\lambda}_f(\mathbf{I_{fmax}} - \mathbf{I_f}) \tag{5.9}$$

The first order conditions of interest are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} - \widetilde{\boldsymbol{\mu}}_h[Y_h][e^{j\theta}] = \mathbf{0} \tag{5.10}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_f}} = -\frac{\partial FV(\mathbf{I_f})}{\partial \mathbf{I_f}} + \widetilde{\boldsymbol{\mu}}_h[e^{j\psi}] - \boldsymbol{\lambda}_f = \mathbf{0} \tag{5.11}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_{fmax}}} = -\frac{\partial FF(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} + \boldsymbol{\lambda}_f = \mathbf{0} \tag{5.12}$$

Equation 5.10, describes the same expression for calculating the marginal prices as was used in Chapter 3. The presence of an active filter in no way changes how the marginal prices are calculated. This is attractive, as the according to equation 5.10, to calculate the prices information is only required on the harmonic voltage distribution through out the network and the loads' preferences. No information about the active filter is required in calculating the prices.

Looking at equation 5.11 and considering the case where the constraint is not binding, the result is:

$$\frac{\partial FV(\mathbf{I_f})}{\partial \mathbf{I_f}} = \widetilde{\boldsymbol{\mu}}_h[e^{j\psi}] \tag{5.13}$$

Given that $\frac{\partial FV(\mathbf{I_f})}{\partial \mathbf{I_f}}$ is a real number, $\widetilde{\boldsymbol{\mu}}_h[e^{j\psi}]$ must also be a real number. The implication being:

$$\psi_i = -\angle\widetilde{\mu}_{hi} \quad \forall\, i \tag{5.14}$$

Equation 5.14 makes sense, stating that the angle of current injection from an active filter will be that which yeilds the owner of the filter the largest payoff. This angle will be in phase, or π radians out of phase with the prevailing angle of current injection, depending on how loads value harmonic distortion. As demonstrated in Section 3.3, equation 5.10 can manipulated to show:

$$\widetilde{\mu}_{hi} = e^{-j\alpha} \sum_{t=1}^{n} y_{ti}^{-1} \frac{\partial u_t(V_{ht})}{\partial V_{ht}}$$

Note that in Section 3.3 α was the common angle of harmonic current injection. Here no such restriction is placed on the angle of harmonic injections. As such α represents an the equivalent common angle of injection, for the rest of the network. The angle of the price described above will be dependent on $\frac{\partial u_t(V_{ht})}{\partial V_{ht}}$. To this point it has been assumed that loads receive negative utility from voltage distortion $\Rightarrow \frac{\partial u_t(V_{ht})}{\partial V_{ht}} < 0\ \forall t$. When this is the case the marginal price can be rewritten as:

$$\widetilde{\mu}_{hi} = e^{-j(\alpha+\pi)} \sum_{t=1}^{n} y_{ti}^{-1} \left| \frac{\partial u_t(V_{ht})}{\partial V_{ht}} \right| \tag{5.15}$$

$$\therefore \text{ When } \frac{\partial u_t(V_{ht})}{\partial V_{ht}}\ \forall t < 0 \qquad \Rightarrow \qquad \psi_i = \alpha + \pi \qquad \forall i$$

This intuitively makes sense as it says in the case where loads dislike harmonic voltage distortion, the prices will provide active filters an incentive to inject current into the network, π radians out of phase. Or in other words, when harmonic distortion imposes a cost on the network, active filters will maximise their revenue by minimising the voltage distortion throughout the network. This is an example of how marginal pricing acts to encourage behaviour, which improves the welfare of the aggregate network.

On the other hand should harmonic distortion result in positive utility for each of the loads:

$$\text{Where } \frac{\partial u_t(V_{ht})}{\partial V_{ht}} > 0\ \forall t \qquad \Rightarrow \qquad \psi_i = \alpha \qquad \forall i$$

That is should loads benefit from harmonic distortion (an silly notion, but considered for completeness) the optimal solution has active filters injecting harmonic current into the network in phase with the prevailing injections. That is, there exists an incentive for loads to maximise the harmonic distortion throughout the network, which is consistent with utility maximisation where $\frac{\partial u_t(V_{ht})}{\partial V_{ht}} > 0\ \forall t$. The different load current, filter current and marginal price angles, for the two different cases are shown in Figures 5.2 and 5.3.

Apart from information as to what is the optimal phase for active filter injections, equation 5.13, provides information as to the optimal magnitude of current injections. It states an active filter at each busbar should make injections into the network until the marginal cost of making injections equals the marginal payment received for the injections (the price magnitude μ_h). This too is what one would intuitively expect from the network optimality condition.

As mentioned previously the marginal costs for active filter injections are likely to be close to zero, and hence the capacity constraint on the active filter is likely to be binding. When this is the case equations 5.11 and 5.12 can be combined to describe the optimal operating state of the active filter(s).

$$\begin{array}{ccccc} \widetilde{\mu}_h[e^{j\psi}] & = & \dfrac{\partial FV(\mathbf{I_f})}{\partial \mathbf{I_f}} & + & \dfrac{\partial FF(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} \\ \text{Marginal payment} & = & \text{Marginal cost} & + & \text{Marginal cost of capital} \\ \text{for injection} & & \text{of injection} & & \text{associated with the injection} \end{array} \tag{5.16}$$

Given marginal operating costs are likely to be approximately zero, equation 5.16 states that filter capacity should continue to be added until the marginal cost of that filter capacity exceeds the value of the harmonic injections (as described by $\widetilde{\mu}_h$). The general solution to the active filter allocation problem (equation 5.16), suggested that filter capacity can be present at each busbar. In the case of a strong network where there is a declining marginal capital cost for

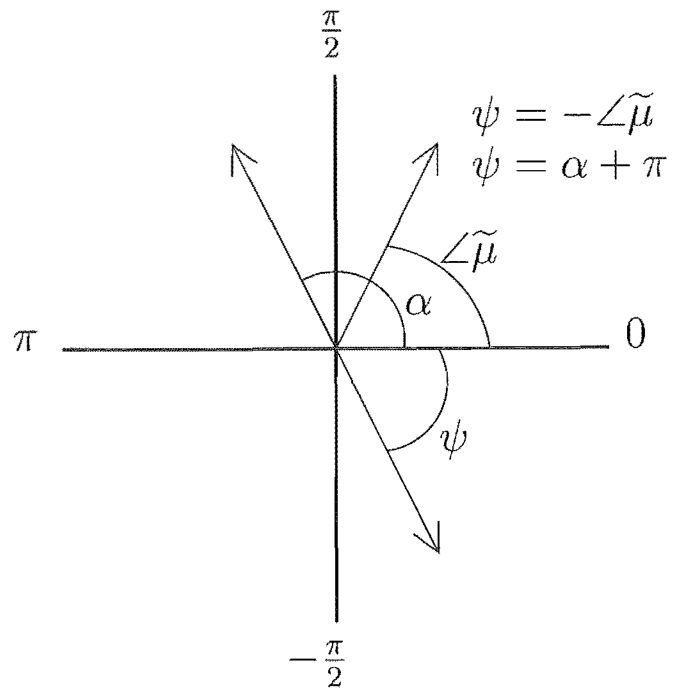


Figure 5.2 Filter current, load current and marginal price angles, where $\frac{\partial u_t(V_{ht})}{\partial V_{ht}} < 0 \; \forall t$

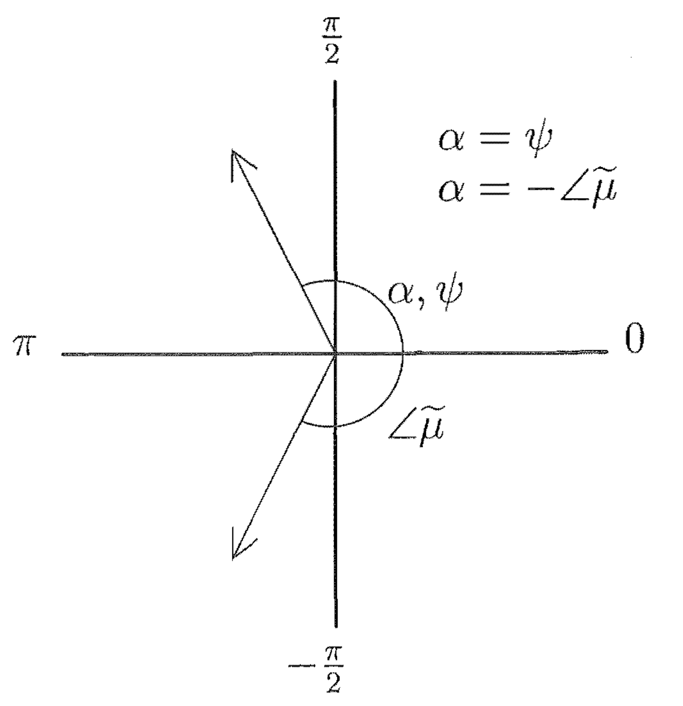


Figure 5.3 Filter current, load current and marginal price angles, where $\frac{\partial u_t(V_{ht})}{\partial V_{ht}} > 0 \; \forall t$

capacity, this optimal solution will consist of single filter at some preferred busbar. The capacity of this filter should be increased to the point where the harmonic marginal price reduces to meet the marginal capital cost of filter capacity. In the case of linear utility functions, this means installing a filter of sufficient size to reduce the harmonic voltage through out the network to zero. Of course should the marginal capital cost of filter capacity always exceed the value of the injections, the optimal solution is to not install any capacity. Simply put the above result shows the optimal filter for each busbar in the network is one whose marginal cost per period, equals the marginal value of the distortion reduction provided (equation 5.17).

$$\widetilde{\mu}_{hi}e^{j\psi_i} = \frac{\partial FV_i(I_{fi})}{\partial I_{fi}} + \frac{\partial FF(I_{fimax})}{\partial I_{fimax}} \qquad \forall\, i \tag{5.17}$$

The solution to the filter resource allocation problem above is a single period solution. As mentioned, though the decision on how much filter capacity to install may have implications beyond a single period. Only a single filter may need to be added to the system for many years. Hence the decision regarding the size of the filter may determine what the filter current will be constrained to over a number of years. Having said that, if active filters were cheap and easily installed into the network the dynamic model may not be necessary. In this case the amount of filter capacity could be quickly adjusted, and the amount of capacity installed during one period need not act as a constraint during other periods. If filter capacity is easily adjusted, the single period solution given by equation 5.17 describes the optimal network conditions. Next the case where the filter decision is a multi-period problem is examined. The internalised problem, which gives the optimal level of filter investment for the network is:

$$\text{Maximise} \quad U(\underline{\mathbf{V_h}}) - FVD(\underline{\mathbf{I_f}}) - FFD(\mathbf{I_{fmax}}) \tag{5.18}$$

$$\text{Subject to} \quad \underline{\widetilde{\mathbf{I}}_\mathbf{h}} + \underline{\widetilde{\mathbf{I}}_\mathbf{f}} - [[Y_h]]\,\underline{\widetilde{\mathbf{V}}_\mathbf{h}} = \underline{\mathbf{0}} \tag{5.19}$$

$$\underline{\mathbf{I_f}} \leq \mathbf{I_{fmax}}\underline{\mathbf{1}} \tag{5.20}$$

Where $FVD(\underline{\mathbf{I_f}})$ and $FFD(\mathbf{I_{fmax}})$ are the multiperiod fixed and variable filter costs, which are a function of the injected filter current and the installed capacity. This optimisation problem extends over an unspecified number of pricing periods. To represent the different states of the network during each time period, the voltage and current vectors have been altered so that information is contained for each period under consideration. If any decision can not be altered for T periods, then the voltage vector $\underline{\widetilde{\mathbf{V}}_\mathbf{h}}$, will contain the voltage at each of the n busbars, for each of the T periods. Previously the voltages throughout the network were given by:

$$\widetilde{\mathbf{V}}_\mathbf{h} = \begin{pmatrix} \widetilde{V}_{h1} \\ \widetilde{V}_{h2} \\ \vdots \\ \widetilde{V}_{hn} \end{pmatrix}$$

To distinguish between the different time periods, the vector of busbar voltages for period t, is denoted by $\widetilde{\mathbf{V}}_\mathbf{ht}$, and the vector of voltage for all periods is constructed so that:

$$\underline{\widetilde{\mathbf{V}}_\mathbf{h}} = \begin{pmatrix} \widetilde{\mathbf{V}}_\mathbf{h1} \\ \widetilde{\mathbf{V}}_\mathbf{h2} \\ \vdots \\ \widetilde{\mathbf{V}}_\mathbf{hT} \end{pmatrix}$$

The network nodal equation constraint when expanded, takes the form shown in equation 5.21.

$$\begin{pmatrix} \widetilde{\mathbf{I}}_{\mathbf{h1}} \\ \widetilde{\mathbf{I}}_{\mathbf{h2}} \\ \vdots \\ \widetilde{\mathbf{I}}_{\mathbf{hT}} \end{pmatrix} + \begin{pmatrix} \widetilde{\mathbf{I}}_{\mathbf{f1}} \\ \widetilde{\mathbf{I}}_{\mathbf{f2}} \\ \vdots \\ \widetilde{\mathbf{I}}_{\mathbf{fT}} \end{pmatrix} - \begin{bmatrix} [Y_{h1}] & 0 & \cdots & \\ 0 & [Y_{h2}] & & \\ \vdots & & \ddots & \\ & & & [Y_{hT}] \end{bmatrix} \begin{pmatrix} \widetilde{\mathbf{V}}_{\mathbf{h1}} \\ \widetilde{\mathbf{V}}_{\mathbf{h2}} \\ \vdots \\ \widetilde{\mathbf{V}}_{\mathbf{hT}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.21}$$

(Bold symbols represent a spacial vector, showing the values at each of the different busbars. Underlined varaibles are temporal vectors showing values for each of the T different pricing periods. Variables both bold and underlined are vectors, which contain both spacial and temporal information)

The first order conditions for this multi-period optimisation problem look very similar to those of the single period problem, i.e.

$$\frac{\partial \mathcal{L}}{\partial \underline{\mathbf{V_h}}} = \frac{\partial U(\underline{\mathbf{V_h}})}{\partial \underline{\mathbf{V_h}}} - \underline{\widetilde{\mu_h}}\,[[Y_h]]\,[[e^{j\underline{\theta}}]] = \underline{0} \tag{5.22}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{\mathbf{I_f}}} = -\frac{\partial FVD(\underline{\mathbf{I_f}})}{\partial \underline{\mathbf{I_f}}} + \underline{\widetilde{\mu_h}}[[e^{j\underline{\psi}}]] - \underline{\lambda_f} = \underline{0} \tag{5.23}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_{fmax}}} = -\frac{\partial FFD(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} + \underline{\lambda_f[I]} = \mathbf{0} \tag{5.24}$$

Equation 5.22 defines the optimal marginal prices for each of the T periods. For each period the prices given by equation 5.22, are identical to the single period solution in equation 5.10. This is important as the value of marginal pricing is diminished, if multiple time periods need to be considered in the calculation of the prices. The prices contain valuable information as to the value the network places on harmonic injections, calculation of these values is not feasible, if knowledge of the intended actions of each load, and the state of technology for some unspecified period into the future is required. It is shown in Section 5.2.3, that these prices encourage optimal behaviour over the course of time from all loads. This implies, by only using the information about the present harmonic state of the network and load preferences, it is possible to encourage each load to act in an optimal fashion, over an extended time horizon.

Again in a similar result as before, equation 5.23, states that the marginal price during each period should equal the marginal cost of the injections, plus an extra amount if the filter injections are constrained by installed capacity. Using the previous result $\angle\widetilde{\mu}_{ht} = -\psi_t$, and the fact marginal injections costs are minimal compared to capital costs equation 5.23 can simplified to:

$$\begin{aligned} \underline{\mu_h} &= \frac{\partial FVD(\underline{\mathbf{I_f}})}{\partial \underline{\mathbf{I_f}}} + \underline{\lambda_f} \\ &\approx \underline{\lambda_f} \end{aligned} \tag{5.25}$$

Over the period of consideration, the summation of the constraint terms at each busbar, should equal the marginal cost of filter capacity installed at that busbar. Equation 5.24 can be restated in a way that makes this apparent.

$$\sum_i^T \lambda_{fi} = \frac{\partial FFD(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} \tag{5.26}$$

Using equation 5.26, equation 5.25, states that in optimality filter capacity should be added to each busbar, until the marginal cost of filter capital is equal to the summation of the difference between injection prices and the marginal operating costs. Simply stated filter capacity should be added until the marginal cost of capacity over the life of the filter, equals the level of payments received after paying for the marginal injection costs. Figure 5.4, depicts the price duration curve for a filter at busbar i. In this diagram the shaded area must be equal to element i in $\sum_j^T \lambda_{fj}$. The excess of price over marginal costs must equal the cost of marginal capital.

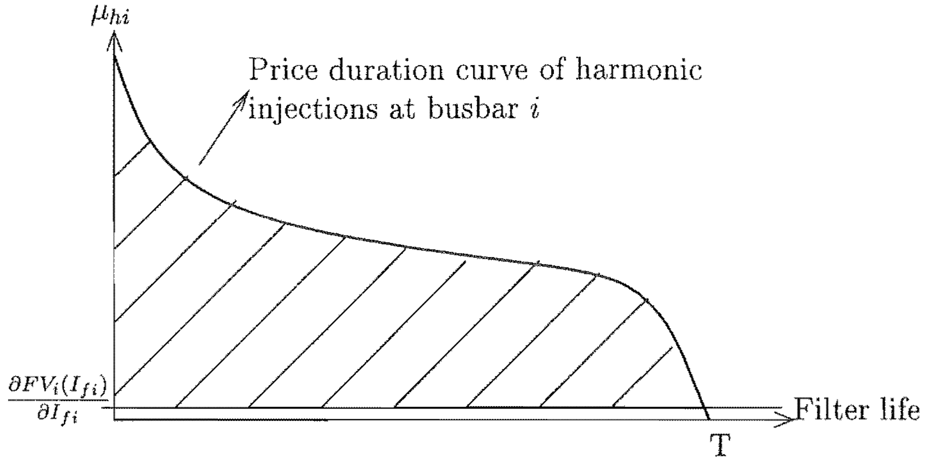


Figure 5.4 Active filter resource allocation

Figure 5.4, shows how over the life of the filter the price should exceed the marginal operating costs, so that the fixed costs associated with the filter are covered.

5.2.3 Individual Optimum

Having looked at the active filter conditions for the network as a whole, here the single period decision making process, for an individual who is considering installing active filter capacity at busbar i, is investigated. Given price $\widetilde{\mu}_{hi}$, for harmonic injections at busbar i, the problem they face is:

$$\begin{aligned} &\text{Maximise} \quad \widetilde{\mu}_{hi}\widetilde{I}_{fi} - FV_i(I_{fi}) - FF_i(I_{fimax}) && (5.27)\\ &\text{Subject to} \quad I_{fi} \leq I_{fimax} && (5.28) \end{aligned}$$

This can be solved using the Lagrangian:

$$\mathcal{L} = \widetilde{\mu}_{hi}\widetilde{I}_{fi} - FV_i(I_{fi}) - FF_i(I_{fimax}) + \lambda_i(I_{fimax} - I_{fi}) \tag{5.29}$$

The first order conditions for this problem are given below.

$$\frac{\partial \mathcal{L}}{\partial I_{fi}} = \widetilde{\mu}_{hi}e^{j\psi_i} - \frac{\partial FV_i(I_{fi})}{\partial I_{fi}} - \lambda_i = 0 \tag{5.30}$$

$$\frac{\partial \mathcal{L}}{\partial I_{fimax}} = -\frac{\partial FF_i(I_{fimax})}{\partial I_{fimax}} + \lambda_i = 0 \tag{5.31}$$

Equations 5.30 and 5.31 can be combined to produce exactly the same condition as the network optimum shown in equation 5.17

$$\widetilde{\mu}_{hi}e^{j\psi_i} = \frac{\partial FV_i(I_{fi})}{\partial I_{fi}} + \frac{\partial FF(I_{fimax})}{\partial I_{fimax}} \tag{5.32}$$

The same conclusions can be drawn from equation 5.32, as were drawn for the network previously. The optimal phase angle active filter current is $\psi_i = -\angle\widetilde{\mu}_{hi}$. Moreover equation 5.32 states that the optimal unconstrained filter injections are those that drive the price down to the marginal operating cost ($\frac{\partial FV_i(I_{fi})}{\partial I_{fi}}$). In the constrained case, prices over the life of the filter should be such that both the operating and capital costs are covered.

While equation 5.32, demonstrates that marginal pricing should encourage efficient decisions over a single period, the problem as posed is not that confronted by the individual loads. As shown in Chapter 4, $\widetilde{\mu}_{hi}\widetilde{I}_{hi}$ will in general be complex, and it is only the real part that is of importance, and hence needs to be charged. The concept of maximising a complex number can be a bit ambiguous, and mis-leading if loads are in reality only charged the real component of $\widetilde{\mu}_{hi}\widetilde{I}_{fi}$. (The case was detailed, as it provides tidy results). To demonstrate that optimal single period decisions will result if loads are only charged the real part $\widetilde{\mu}_{hi}\widetilde{I}_{fi}$, the decision making process for a load i, is detailed below. The problem this load at busbar i, faces can be expressed as:

$$\begin{aligned} &\text{Maximise} \quad Re\{\widetilde{\mu}_{hi}\widetilde{I}_{fi}\} - FV_i(I_{fi}) - FF_i(I_{fimax}) && (5.33)\\ &\text{Subject to} \quad I_{fi} \leq I_{fimax} && (5.34) \end{aligned}$$

This can be solved using the Lagrangian:

$$\begin{aligned} \mathcal{L} &= Re\{\widetilde{\mu}_{hi}\widetilde{I}_{fi}\} - FV_i(I_{fi}) - FF_i(I_{fimax}) + \lambda_i(I_{fimax} - I_{fi}) \\ &= \mu_{hi}I_{fi}\cos(\angle\widetilde{\mu}_{hi} + \psi_i) - FV_i(I_{fi}) - FF_i(I_{fimax}) + \lambda_i(I_{fimax} - I_{fi}) \end{aligned} \tag{5.35}$$

The first order conditions for this problem are:

$$\frac{\partial \mathcal{L}}{\partial I_{fi}} = \mu_{hi}\cos(\angle\widetilde{\mu}_{hi} + \psi_i) - \frac{\partial FV_i(I_{fi})}{\partial I_{fi}} - \lambda_i = 0 \tag{5.36}$$

$$\frac{\partial \mathcal{L}}{\partial I_{fimax}} = -\frac{\partial FF_i(I_{fimax})}{\partial I_{fimax}} + \lambda_i = 0 \tag{5.37}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = -\mu_{hi}I_{fi}\sin(\angle\widetilde{\mu}_{hi} + \psi_i) = 0 \tag{5.38}$$

Equation 5.38 has multiple solutions at $\psi_i = -\angle\widetilde{\mu}_{hi} + n\pi$, where n is an interger. But given that

$$\frac{\partial^2 \mathcal{L}}{\partial I^2_{fimax}} = -\mu_{hi}I_{fi}\cos(\angle\widetilde{\mu}_{hi} + \psi_i) < 0 \qquad \text{for } \psi_i = -\angle\widetilde{\mu_{hi}}$$

The optimal angle of injected currrent from the active filter is $\psi_i = -\angle\widetilde{\mu}_{hi}$, this is consistent with equation 5.17. Substituting this result into equation 5.36 and 5.37 the result is:

$$\mu_{hi} = \frac{\partial FV_i(I_{fi})}{\partial I_{fi}} + \frac{\partial FF_i(I_{fimax})}{\partial I_{fimax}} \tag{5.39}$$

Again this result is consistent with equation 5.17, demonstrating if loads only pay the real part of the harmonic charges, optimal active filter investment decisions will be made.

But as was the case for the network, the individual decisions made with respect to installation of active filters, are likely to have consequences for a number of periods. Like before the active filter allocation problem is a dynamic optimisation problem. It can be described as:

$$\text{Maximise} \quad E[\underline{\widetilde{\mu}_{hi}}]\underline{\widetilde{I}_{fi}} - FVD_i(\underline{I_{fi}}) - FFD_i(I_{fimax}) \tag{5.40}$$

$$\text{Subject to} \quad \underline{I_{fi}} \leq I_{fimax}\underline{1} \tag{5.41}$$

$$\text{Where} \quad \underline{\widetilde{I}_{fi}} = \begin{pmatrix} \widetilde{I}_{fi1} \\ \widetilde{I}_{fi2} \\ \vdots \\ \widetilde{I}_{fiT} \end{pmatrix}$$

$=$ Vector of filter injections at busbar i over time

$E[\underline{\widetilde{\mu}_{hi}}] =$ Vector of expected prices for harmonic injections at busbart i over time

This problem can be solved using the Lagrangian:

$$\mathcal{L} = E[\underline{\widetilde{\mu}_{hi}}]\underline{\widetilde{I}_{fi}} - FVD_i(\underline{I_{fi}}) - FFD_i(I_{fimax}) + \underline{\lambda_i}(I_{fimax}\underline{1} - \underline{I_{fi}}) \tag{5.42}$$

The first order conditions for this Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial \underline{I_{fi}}} = E[\underline{\widetilde{\mu}_{hi}}][e^{j\underline{\psi_i}}] - \frac{\partial FVD_i(\underline{I_{fi}})}{\partial \underline{I_{fi}}} - \underline{\lambda_i} = \underline{0} \tag{5.43}$$

$$\frac{\partial \mathcal{L}}{\partial I_{fimax}} = -\frac{\partial FFD_i(I_{fimax})}{\partial I_{fimax}} + \sum_j^T \lambda_{ij} = 0 \tag{5.44}$$

Again equation 5.43, specifies the injected angle of filter current into the network for each time period. Incorporating the fact that injected current at each period will have the opposite angle to the price (implied in equation 5.43, and shown previously), equation 5.43 can be restated as:

$$E[\underline{\mu_{hi}}] = \frac{\partial FVD_i(\underline{I_{fi}})}{\partial \underline{I_{fi}}} + \underline{\lambda_i} \approx \underline{\lambda_i} \tag{5.45}$$

Equation 5.45, reflects the fact that the marginal operating costs of the filter are likely to be small compared with the cost of capital. Should the expectation of future harmonic prices used in the decision making process, prove to be equal to the actual prices that eventuate, this equation is identical to equation 5.25 (if decomposed into optimal conditions at each busbar). That is should some load i, have accurate expectations as to the future harmonic prices, they shoud behave in a way that is optimal for the network at large.

The obvious question is, how likely are the expectations about future prices, to prove reasonably accurate, and what are the consequences if they are not? The second part of that question is easiest to answer. Should an individual under estimate the harmonic price over the T time periods (the price representing the value of harmonic injections to the network), they will commit a sub-optimal amount of capital towards installing active filter capacity at any busbar. On the other hand any individual whom over estimates the future harmonic prices will be inclined to install excess capacity at a given busbar. This is a situation very similar to the classic "winner curse" that exists with tender auctions of goods with a common value. That is:

When an item to be tendered has an uncertain but common value to all bidding, the individual who bids the highest price will win the tender. In the case where each individuals bidding have reasonable expectations of the true value of the tender item, the mean expected value of the tender item amongst all the bidders, will be equal to its true value. As such the hightest bid that wins the tender must over estimate the items true value. Therefore he who wins must do so because they overpaid.

The value of an installed filter will have a common value, as whoever owns it will receive the same payments for its current injections, and will face the same operating and capital costs. The decision to build a filter is likely to mimic that of an auction of a single item, as in the case of a strong network there are likely to be limited number of filters present, as any single filter will influence the distortion present at all busbars. In a weak system this is not the case, but here the consequences of any harmonic injections are felt mainly at the busbar where they occur, therefore all the externalities are internalised, and optimal individual behaviour does not require guidance in the form of marginal prices. The consequence of this is those who add filter capacity to the network, are those who over estimate the value of the filters injections, and these individuals are likely to install a greater than optimal level of filter capacity. So long as the trend of increasing harmonic injections into the network continues, this capacity will eventually be utilised, but by that time there is likely to have been advances in technology, which will provide a cheaper solution to the problem.

The extent to which this over investment in filter resources takes place, will be dependent on how accurately individuals can predict the future value of harmonic injections. It is hard to make any statements about this, apart from anecdotal observations, which lead the author to believe individual's belief in their ability to forecast the future, tends to be far in excess of their actual abilities. The result being, individuals fail to appropriately discount the values they place on these items, due to the fact they are rather confident of their predictions. The winners curse would not exist if all bidding accepted the fact they were uncertain as to the item's true value, and discounted their bid accordingly.

The possibility that the expected future value of injections may not be accurate, does not leave marginal pricing as redundant, as to achieve an efficient allocation of resources some effort towards establishing this value must be taken. A decision to use other techniques on the basis that the exact future value of distortion cannot be accurately established would be illogical, as it suggests it is preferable to be certainly wrong, than possibly correct. Also financial disappointment will come to those less skilled towards predicting the future value of harmonic distortion, leading these individuals away from making such decisions, and an improvement in accuracy of decision making over time. As the consequences of this uncertainty, is an excess of filter capacity above the optimal level, those with a conservative engineering approach to such problems should not be too troubled by this result.

Individual Incentive

If investing in filter capacity when others are not implies that the decision is not wise, why would any rational individual do so? The profit maximisation problem as described above, may not match reality, as the installation of filter capacity might be used to deter others from entering the market (ie. installing competing capacity). It is also likely the active filter will have considerable influence over the marginal harmonic price throughout the network. The previous work implicitly assumed the owner of the filter had little ability to control the harmonic price. This assumption is valid so long as any filter owner is unable to prevent other individuals from installing filter capacity, if economic to do so. But in reality spare filter capacity can be used to deter others from installing filters, as the incumbent has the potential to drive the price for harmonics down to zero in the event another individual enters the market. This situation is common in commodity industries [Varian1996]. To demonstrate this the decision making process for an individual is detailed again, where the expected harmonic price is defined as a function of the injected current of the filter, along with the amount of filter capacity installed. This revised profit maximisation problem then is:

$$\text{Maximise} \quad E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]\underline{\widetilde{I}_{fi}} - FVD_i(\underline{I_{fi}}) - FFD_i(I_{fimax}) \tag{5.46}$$

$$\text{Subject to} \quad \underline{I_{fi}} \leq I_{fimax}\underline{1} \tag{5.47}$$

$$\text{where} \quad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{5.48}$$

This problem can be solved using the Lagrangian

$$\mathcal{L} = E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]\underline{\widetilde{I}_{fi}} - FVD_i(\underline{I_{fi}}) - FFD_i(I_{fimax}) + \underline{\lambda_i}(I_{fimax}\underline{1} - \underline{I_{fi}}) \tag{5.49}$$

The first order conditions for this Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial \underline{I_{fi}}} = \frac{\partial E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]}{\partial \underline{I_{fi}}}\underline{\widetilde{I}_{fi}} + E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})][e^{j\underline{\psi_i}}] - \frac{\partial FVD_i(\underline{I_{fi}})}{\partial \underline{I_{fi}}} - \underline{\lambda_i} = \underline{0} \tag{5.50}$$

$$\frac{\partial \mathcal{L}}{\partial I_{fimax}} = \frac{\partial E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]}{\partial I_{fimax}}\underline{\widetilde{I}_{fi}} - \frac{\partial FFD_i(I_{fimax})}{\partial I_{fimax}} + \sum_{j}^{T} \lambda_{ij} = 0 \tag{5.51}$$

The conditions detailed in equations 5.50 and 5.51, clearly no longer match those of the network optimum detailed in 5.25. These expressions can be simplified to remove reference to the angle of filter injections, or the angle of the harmonic price.

$$E[\underline{\mu_{hi}}(\underline{I_{fi}}, I_{fimax})] = -\frac{\partial E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]}{\partial \underline{I_{fi}}}\underline{\widetilde{I}_{fi}} + \frac{\partial FVD_i(\underline{I_{fi}})}{\partial \underline{I_{fi}}} + \underline{\lambda_i}$$

$$\begin{matrix} \text{Marginal income from} \\ \text{filter injections} \end{matrix} = \begin{matrix} \text{Marginal lost revenue due} \\ \text{to price reductions} \end{matrix} + \begin{matrix} \text{Marginal cost of} \\ \text{filter injections} \end{matrix} + \begin{matrix} \text{Marginal cost of} \\ \text{capital constraint} \end{matrix} \tag{5.52}$$

Similarly the value of the constraint terms differ, to reflect the fact the filter capacity has value in its abilityto deter filter investment by others, which pushes up the price.

$$\sum_{j}^{T} \lambda_{ij} = \frac{\partial FFD_i(I_{fimax})}{\partial I_{fimax}} - \frac{\partial E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]}{\partial I_{fimax}}\underline{\widetilde{I}_{fi}}$$

$$\begin{matrix} \text{Sumation of} \\ \text{constraint terms} \end{matrix} = \begin{matrix} \text{Capital cost of marginal} \\ \text{unit of filter capacity} \end{matrix} - \begin{matrix} \text{Marginal increase in revenue} \\ \text{due to marginal filter capacity} \end{matrix} \tag{5.53}$$

Equation 5.52, is identical to the optimal conditions described earlier except for the extra term $-\frac{\partial E[\underline{\widetilde{\mu}_{hi}}(\underline{I_{fi}}, I_{fimax})]}{\partial \underline{I_{fi}}}\underline{\widetilde{I}_{fi}}$. This term describes the reduction in revenue that results from a marginal increase in filter injections, due to the effect those injections have on the prevailing price. This demonstrates how when in a position of market power, it may be optimal for the owner of the filter to withhold injections from the network, as the increased voltage distortion levels throughout the network, will increase the value of the injections the active filter does make. It should be noted that the constraint term in this case also has a different meaning. Previously the constraint was related solely to the cost of capital, of the marginal unit of filter capacity. In this case the value of the constraint is reduced, as the filter capacity now posses value, in that it acts as a deterrent to other individuals adding capacity to the network. The value of this capacity is captured by the term $\frac{\partial E[\widetilde{\mu}_{hi}(I_{fi},I_{fimax})]}{\partial I_{fimax}}\underline{\widetilde{I}_{fi}}$. The fact that the constraint term is reduced will act to reduce the extent to which the monopoly filter will boost prices as compared to before, ie. some of the price rise resulting form the $-\frac{\partial E[\widetilde{\mu}_{hi}(I_{fi},I_{fimax})]}{\partial I_{fi}}\underline{\widetilde{I}_{fi}}$ term will in fact be contributing towards the cost of capital as now $\sum_j^T \lambda_{ij} < \frac{\partial FFD_i(I_{fimax})}{\partial I_{fimax}}$. But the extent to which the term $-\frac{\partial E[\widetilde{\mu}_{hi}(I_{fi},I_{fimax})]}{\partial I_{fi}}\underline{\widetilde{I}_{fi}}$, will pick up the capital costs will be limited, as it is only spare filter capacity above the optimal amount that is able to provide any sort of deterrence to others adding capacity, ie $\frac{\partial E[\widetilde{\mu}_{hi}(I_{fi},I_{fimax})]}{\partial I_{fimax}} \to 0$ for I_{fimax} less than that which is optimal.

It is hard to make any definitive conclusions as to how an individual would behave in such a situation as so much is dependent of the cost structure of filter technology (minimum efficient scale in particular); of which this work makes no assumptions. But it is clear there is the potential incentive for a filter owner to hold back filter injections so as to profit at the expense of the network. This may not be a problem, in that active filters are designed to minimise the voltage distortion seen at the filter busbar. The re-engineering of any active filter to maximise income may prove to be an expensive task, making any such plans uneconomic. Another option is rules can be imposed stipulating that any filter capacity installed in the network must operate so to minimise the voltage distortion present up to the full capacity of the filter. There are property right issues associated with such a rule, which may not make it attractive (and which are not discussed here), but it would work. Finally, filter investment could be made by an entity that has no interest at profiting from the harmonic market. It is likely for the foreseeable future a harmonic market will not be formed, and network infrastructure companies will own filters to provide a given level of service quality. In this situaton marginal pricing techniques can still be used by the network operator as a guide as to the optimal level of filter investment. In these circumstances where there is no potential to profit from a harmonic market, the marginal prices will provide valuable information leading to an efficient level of investment, the costs of which can be distributed fairly.

5.3 ACTIVE FILTER EXAMPLE

Using the test system detailed in Appendix A, an example of an active filter within a marginal pricing environment is presented in this section. To keep things simple in this example the single period optimisation framework is used. While not a realistic representation of the decision making process regarding filters, it was shown previously that the results gleaned from this model carry through to the more realistic multi-period framework.

It was also discussed previously that marginal pricing had value in its ability to specify the optimal placement of filters within a network. For the test system used here, the placement of the filter makes little difference to it value. This because all the busbars are closely coupled, and the injections into one busbar have a similar effect at all the others throughout the network. The fact that there is no highly preferred busbar for the placement of a harmonic filter, can be seen in previous example in Section 3.3.1, as the harmonic price at each busbar was approximately equal. In an uncoupled network the prices through out the network will vary to reflect the fact injections at different points have a different value. As such it was decided the for following example an active filter would be located at busbar one (Figure 5.5).

In Section 5.2.3, it was shown there potentially exist incentives for the owner of a filter to withhold injections from the network, to push up the harmonic price. This example assumes that those issues have been dealt with, so that the owner of a filter is unable to manipulate the price. It should be noted though, even with the filter owner taking non-competitive actions to boost the value of their injections, the network is better off in that situation, than in the absence of the filter.

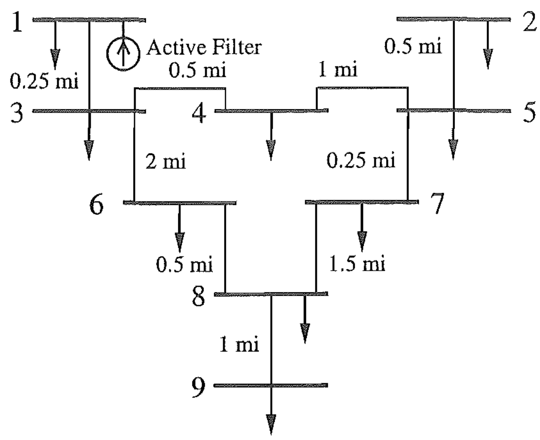


Figure 5.5 Test network with an active filter located at busbar one

For this example it will be assumed that the marginal cost of harmonic injections is negligible. So that the only cost associated with a filter is the cost of capital. It is assumed that cost of capital for the filter at busbar one is a function of the installed capacity squared.

$$\begin{gathered}\text{Cost of capital for filter at busbar one } FF_i(I_{fimax}) = FI_{fimax}^2 \\ \text{Where } F = \$50/A^{\frac{1}{2}}\end{gathered} \tag{5.54}$$

In this example three different cases are detailed:

1. No filter in the system, no action taken by loads to reduce their injections
2. Optimal filter in the system, loads do not respond to harmonic prices
3. Optimal filter in the system, loads respond to harmonic prices if profitable

The loads all have linear harmonic utility functions, and linear costs of harmonic reduction as detailed in Appendix A. As a result, so long as there is some level of harmonic voltage throughout the network the marginal price for harmonic injections is constant with a magnitude approximately equal to $\$35/pu$. Given this constant price, in case three, it is optimal for the loads at busbars three and six to reduce their injections to zero. Moreover due to the constant marginal price the optimal level active filter capacity (and injections) for both cases two and three is constant. This would not be the case if after the harmonic reductions I_{R3} and I_{R6}, the filter injections of case two were greater than that required to reduce the harmonic voltages to zero.

Table 5.1 Resultant harmonic injections and voltages before and after injection reductions

	Harmonic Price μ_{hi} (\$/pu)	Filter Capacity I_{f1max} (pu)	I_{R3} (pu)	I_{R6} (pu)
Case 1	35.40	0.0000	0.00	0.00
Case 2	35.40	0.3545	0.00	0.00
Case 3	35.40	0.3545	0.06	0.30

Figure 5.6 shows the resultant harmonic voltage magnitude throughout the network for the three cases. For each case, the voltage at each busbar is approximately equal, a result of the strong test system. It can be seen that the active filter injections, and the reduction in injections by the loads at busbars two and three, reduce the distortion magnitude by 85%.

Figure 5.7, shows how the harmonic payments made by each load vary for the three cases. As the marginal prices are constant each load pays the same amount, except the loads at busbars three and six, who pay nothing in case three as they reduce their injections to zero.

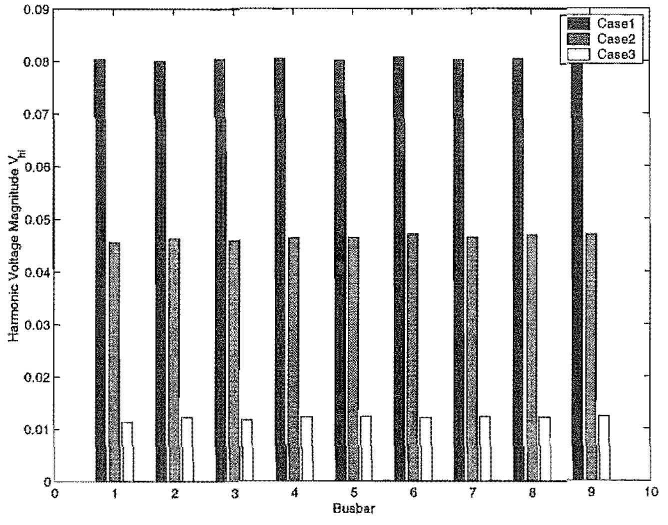


Figure 5.6 The harmonic voltage magnitude at each of the busbars for the three cases

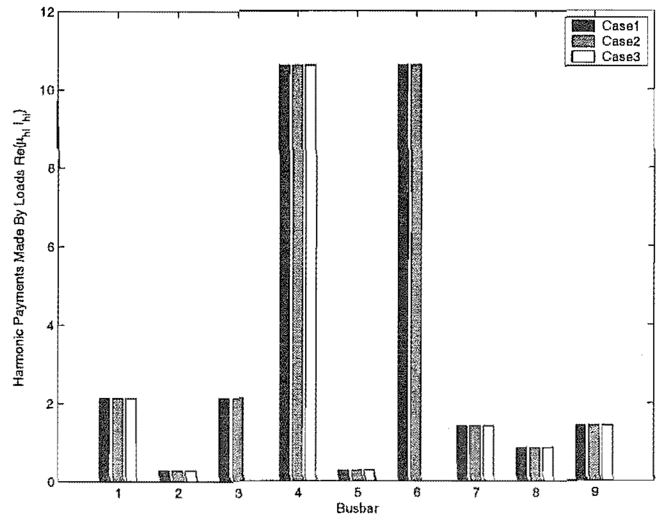


Figure 5.7 The harmonic payments made by each load for the three cases

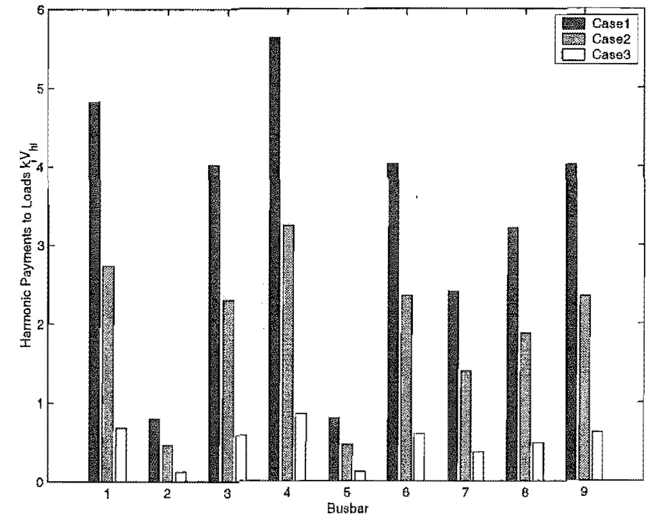


Figure 5.8 The harmonic compensation payments made to each load for the three cases

Figure 5.8, shows how the harmonic compensation payments vary with each case. Given the payment to each load= $k_i V_{hi}$, the graph just represents each loads valuation of voltage distortion, scaled by the distortion magnitude for each case.

Of interest is the effect the marginal pricing has on the aggregate utility of the network, as a result of the signals and incentives provided to loads and potential filter owners. It was put forward earlier that marginal pricing would encourage optimal investment in filter resources, which should result in a corresponding increase in network utility. The aggregate network utility is detailed for each of the cases in Table 5.2, and indeed this is the case.

Table 5.2 Total network utility variation as filter and loads respond to marginal prices

	Case 1	Case 2	Case 3
Total Network utility	−29.76	−23.47	−16.72

The total utility calculated above for each of the three cases was done so using equation 5.55

$$\begin{aligned}\text{Total System Utility} &= \text{Harmonic Voltage Utility} - \text{Cost of Harmonic Reductions} - \text{Cost of Filter}\\ &= \mathbf{KV_h} - \rho\mathbf{I_R} - FI_{fimax}^2\end{aligned} \tag{5.55}$$

Table 5.2, indicates that the system as a whole, benefits from the filter and reduction in injected harmonic current that is signalled as optimal by the marginal prices. Figure 5.9, shows that each individual load is no worse or better off, as a result of the filter injections and reduction in load injections that result from marginal pricing.

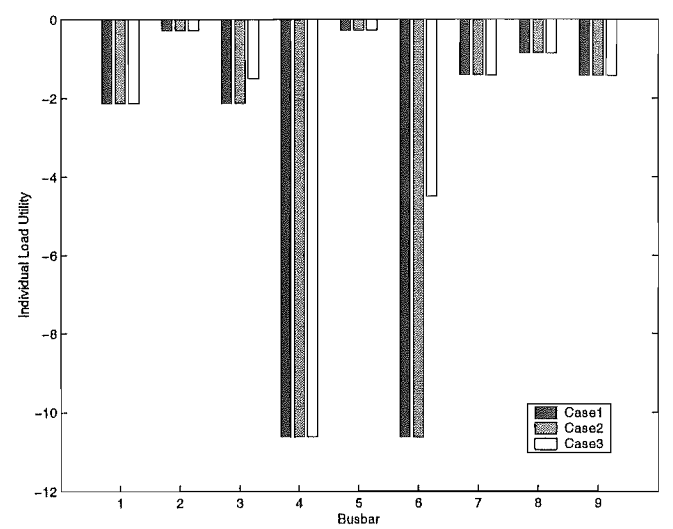


Figure 5.9 Harmonic utility for each load for the three cases

The utility for each of the individual loads is calculated as:

$$\begin{aligned}\text{Utility of Load } i &= -\text{Harmonic Payments By } i - \text{Cost of Any Harmonic Reductions By } i\\ &= -Re\{\widetilde{\mu}_{hi}(\widetilde{I}_{hi} - \widetilde{I}_{Ri})\} - \rho_i I_{Ri}\end{aligned} \tag{5.56}$$

In equation 5.56, load i's utility due to harmonic voltage distortion at the local busbar, and the harmonic payments received are not included as they cancel out.

Having established that the marginal prices should lead to actions, which make the aggregate network and each particular load better off, the final point of interest is to look at how the filter owner fairs under marginal pricing. The filter owner is paid $\widetilde{\mu}_{h1}$, for its injections into the system, while the cost of filter capital required to produce those injections are FI^2_{f1max}. Given the assumptions that the filter owner is not going to attempt to abuse any market power they may posses, it is clearly optimal for the owner to make injections into the network up to filter capacity. As such filter owner utility is given by:

$$\begin{aligned}\text{Utility of Filter Owner} &= \widetilde{\mu}_{h1}\widetilde{I}_{f1max} - FI^2_{f1max} \\ &= 6.28\end{aligned} \tag{5.57}$$

It is evident that this particular example the owner of the filter was clearly advantaged by having the opportunity to sell injections into the network at their marginal value. This example therefore demonstrates how marginal pricing acts to coordinate load and filter owners in the network, so that each acts in a way that improves their own welfare, along with that of the aggregate network.

5.3.1 Toll Road Comparison

Next the allocation of costs that results from marginal pricing are compared to those which would result from the Toll Road methodology (Section 2.3.2). Consider the case where the amount of filter capacity installed is the optimal amount found in cases two and three previously. The cost of filter capacity in this section is identical to that before (equation 5.54).

$$\begin{aligned}&\text{Filter Capacity Installed} = 0.3545pu \\ &\Rightarrow \text{Cost of Filter Capital} = \$6.28\end{aligned}$$

Figure 5.10 shows the net payments made by each load under both Toll Road and Marginal pricing. The net payments in the case of Toll Road pricing, refers to the payments made to pay for the installed filter. In the case of marginal pricing, net payments, refers to the harmonic payments made ($Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}\ \forall i$), less the compensation payments received for the voltage distortion seen at their busbar ($k_iV_{hi}\ \forall i$), the sum of these residual payments equal the payments made to the filter ($\widetilde{\mu}_{h1}\widetilde{I}_{h1}$). At this point the possibilility of loads reducing their injections due to the prices has not been included.

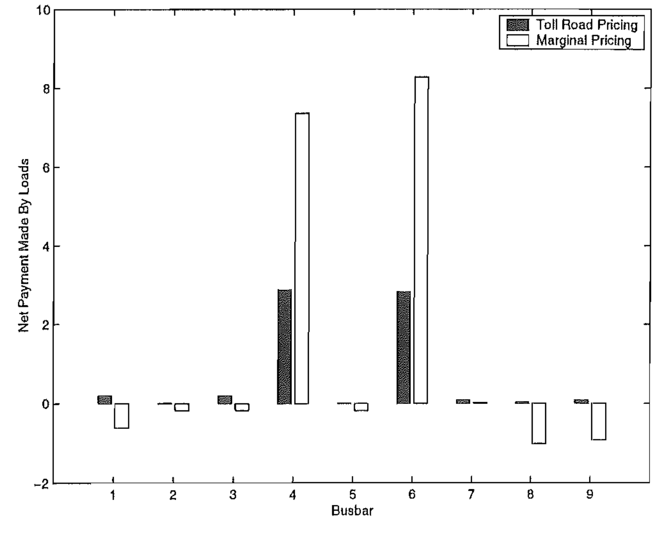


Figure 5.10 Comparison of the net payments made by loads under Toll Road and Marginal Pricing

It can be seen that there are clearly considerable differences between the payments made under the two systems. The most obvious difference is that with marginal pricing, the net payments by some loads are negative, while with Toll Road pricing all loads face positive net payments. This should not be interpreted as saying under marginal pricing some loads are paid for their injections. To the contrary all loads face the same price for their injections ($\widetilde{\mu}_h$), and pay an amount, which reflects their impact on the aggregate system ($Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}\ \forall\, i$). Instead with marginal pricing loads are paid compensation for the voltage distortion seen at their busbar ($k_i V_{hi} \forall\, i$). For some loads these compensation payments are in excess of the harmonic payments made. Under Toll Road pricing no such compensation payments exist.

In the test system used the loads at busbars four and six are responsible for the majority of the harmonic injections, therefore under both pricing systems they have the largest payments to make. With marginal pricing they are required to make considerably larger payments, as not only do filter costs have to be met, but so do the costs their injections impose on all others through out the network.

Note that the payments to the filter differ considerably under Toll Road and marginal pricing. Under marginal pricing the filter is paid the harmonic price for all injections, that is the filter is paid in accordance with the value those injections have to the aggregate network. Toll Road pricing on the other hand makes no consideration of what the filter payments are worth, and instead a set amount is collected from the network that equals the filter cost over the period.

Table 5.3 Total payments to the filter under Toll Road and Marginal Pricing

	Toll Road Pricing	Marginal Pricing
Filter Payments	6.28	12.57

This highlights the greatest weakness in Toll Road pricing, in that it provides no signals as to what level of filter investment is optimal, and no incentive exists for that level of investment to take place. As shown in Table 5.3, under marginal pricing the filter owner receives payments in excess of the costs. These excess payments are maximised at the point where the optimal level of filter capacity is installed, providing the incentive to move towards an efficient level of filter investment. An efficient investment is one where value in excess of the costs is created. With marginal pricing this excess value is available to the filter owner, with Toll Road pricing this excess value, should it exist, is distributed to the loads, and hence there is no incentive or signal as to what is the optimal level of filter investment. If the Toll Road method were to be used it is unlikely that the filter capacity of the previous example would have been selected. Instead with the Toll Road method filter capacity would be installed to reduce the harmonic voltage throughout the network to some specified level, as dictated by regulations or some standard.

The potential difference in network welfare between using Toll Road and marginal pricing, is illustrated in Table 5.4. This shows the likely result of the different pricing systems, if under Toll road pricing filter capacity is added to the point where the resultant distortion throughout the network is approximately zero. The marginal pricing results are the optimal equilibrium shown previously, and the result when no action is taken show the consequences if no filter is added and loads do not reduce their harmonic injections.

Table 5.4 Likely outcomes under Toll Road and Marginal Pricing

	I_{f1max}	Filter Cost ($)	I_{R3}	I_{R6}	Mean Busbar Voltage V_{hi}	System Utility
No Action Taken	0.000	0.00	0.00	0.00	0.080	−29.76
Marginal Pricing	0.355	6.28	0.06	0.30	0.012	−16.72
Toll Road Pricing	0.807	32.56	0.00	0.00	0.003	−33.74

From Table 5.4, one can see that with Toll Road pricing, there is a much larger amount of filter capacity added to the network, in an effort to reducing the voltage distortion towards zero. But the addition of this additional capacity is not economic and the final result is a level of network welfare that is lower than before any action was taken. This example has been constructed so that reducing the distortion level to zero would be expensive and sub-optimal. But until an attempt to measure the value of distortion to each load is made, which in turn easily leads to the calculation of marginal prices for harmonic injections, it is impossible to know if the filter capacity being added is improving the welfare of the system. This may be especially true in the case where filter capacity is added so to meet some predetermined standard. Without going through a process essentially equivalent to calculating marginal pricing, there is no basis via which to estimate what is the optimal level of filter capacity for the network. The marginal prices also have the desired quality that they provide each individual incentives to move towards the optimal point. As in this example, there would clearly be individuals willing to install the optimal amount of filter capacity at busbar one as they are rewarded for doing so in excess of the costs.

Figure 5.11, shows the net payments made by each load under Toll Road and marginal pricing for this example. Previously in Figure 5.10, some loads under marginal pricing had a net cash inflow. This is not the case here. Allowing the loads to reduce their injections (if advantageous), has seen a dramatic reduction in the voltage distortion, reducing the compensation payments received by each load. In the previous example where the filter capacity was fixed at the optimal level and the loads were unresponsive, the two main polluters had the most to gain from Toll Road pricing, as it stopped them from having to compensate all the other loads for the distortion they cause. But as soon as the loads are allowed to respond to the prices, and one includes the fact Toll Road pricing is unlikely to lead to an excess amount of filter capacity, the same loads are the ones most in favour of marginal pricing. This is not surprising as they have the largest stake in the harmonic state of the network and hence stand to be the most penalised if the network is not at an efficient point.

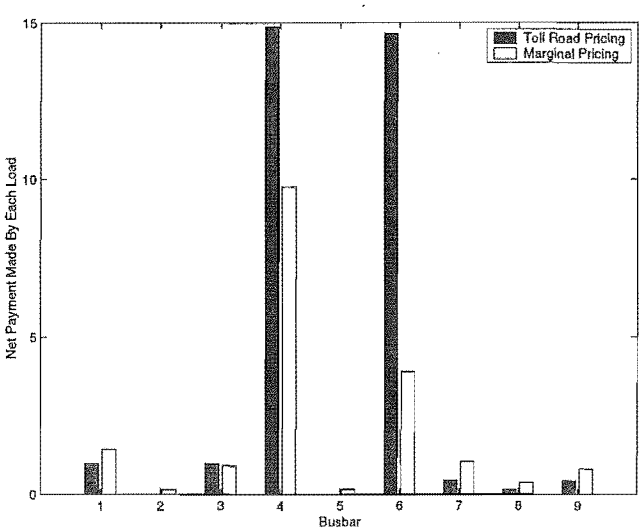


Figure 5.11 Comparison of the likely net payments made by loads under Toll Road and Marginal Pricing

5.3.2 Non-Linear Utility Example

This example is identical to that of Section 5.3.1, except that the utility function of each firm i, has been altered to be dependent on the square of the harmonic voltage been at their busbar

(V_{hi}^2). The utility function for each load has been changed in the same systematic way, as shown in equation 5.58.

$$\text{Utility of load at busbar } i = 50k_i V_{hi}^2 \qquad \forall i \tag{5.58}$$

Each load receives negative utility from harmonic distortion ($k_i < 0 \quad \forall i$). The utility functions as well as being changed to a quadratic function of the voltage magnitude, was also multiplied by 50. This was simply to produce numbers of a similar magnitude to the previous example. Clearly no direct comparison should take place between the two examples.

Table 5.5, shows the likely outcome, in the three situations:

- No action is taken with respect to installation of filters or reducing injections of loads
- Marginal pricing is used, loads and filters react so as to maximise individual utility
- Toll Road pricing used, filter capacity added till harmonic distortion falls close to zero

In Table 5.5, harmonic prices have been given for each of the three cases. Note these prices only officially exist for the second case where the prices are explicitly used. But the prices contain information, in that they specify what value the network places on harmonic injections given its present harmonic state. As such the prices are calculated for each case to indicate what harmonic injections are worth at the margin, given the harmonic state of the network. The results are similar to the previous example in that Toll Road pricing is less efficient than marginal pricing on account of the fact that no signals exist as to what is the optimal level of filter capacity, and reducing distortion to zero proves to be inefficient. Under the Toll Road method the implied marginal cost placed on the injections from the active filter is one forth that implied by marginal pricing. This would not be a problem, except at the Toll Road pricing equilibrium, the marginal cost of filter capacity is twice that of the marginal pricing equilibrium.

Table 5.5 Likely outcomes under Toll Road and Marginal Pricing

	I_{f1max}	Filter Cost (\$)	Busbar Voltage V_{hi}
No Action Taken	0.000	0.00	0.080
Marginal Pricing	0.371	6.86	0.011
Toll Road Pricing	0.808	32.65	0.003
	Harmonic Price μ_{hi}	Injection Reduction I_{Ri}	System Utility
No Action Taken	284.7	nil	−119.6
Marginal Pricing	37.0	$I_{R3}\&I_{R6}$	−14.9
Toll Road Pricing	10.6	nil	−32.8

5.4 CONCLUSION

Clearly to deal with the harmonic distortion efficiently, efficient decisions with respect to allocation of filter resources must be made. It must be determined if the harmonic distortion is serious to the point of needing some action to be taken. Also the correct decision must be made between allocating resources towards filters versus other mitigation methods. Finally information is required as to what is the optimal filter, to meet the needs of the network. In the absence of marginal harmonic prices, which place an explicit value of harmonic injections, it is impossible to achieve the above. The present approach to filter allocation has been to draw a line in the sand and say a certain level of distortion is acceptable, and action must be taken to ensure this level is not exceeded. Clearly this approach cannot achieve the most efficient outcome.

While harmonic prices that signal the value of any potential filter injections, there is an added complexity, in that selection of an efficient filter is a multi-period optimisation problem. This is because filters take time to design and build, meaning the filter capacity in the network can not be instantly altered. Moreover filters have a lifespan that is likely to cover many periods of differing harmonic prices. The dynamic nature of the optimisation problem is easily accounted for and the results are essentially unchanged from the case where the problem is simplified and modelled as a static optimisation problem.

The conclusions were that in optimality, filter capacity should be added to the point where the marginal price for injections equals the marginal operating cost for the injection, plus over time an amount equal to the cost of capital for the marginal filter capacity. The result being that over the operating life of the filter both its operating and capital costs are covered. There is no great surprise with this result, but in practice achieving such a result is not a simple task. It was found though, the existence of marginal prices that correctly signal the value the network places on harmonic injections, provide an incentive for individuals to install an optimal amount of filter capacity. While incentives exist for individuals to act in an optimal manner, their action will be based on expectations of future marginal prices. It is unlikely that these expectations will prove to be completely accurate, especially if it is considered that the expectations cover the whole period of the filter's life. However this requirement does not make marginal pricing less effective, in that no matter how one deals with harmonics there is a requirement for a certain amount of forecasting. To suggest this can not be done accurately in a marginal pricing environment, also suggests the same for other approaches. Hence this requirement, that for optimality to be achieved individuals must be able to accurately forsee the future, makes marginal pricing no more or less likely to achieve an efficient outcome than other approaches. As mentioned though, a likely outcome is an excess amount of filter capacity installed in the network. The cost burden of which will fall on those who made the decision to install it. This meets the fairness requirement in that loads are not held liable for inefficient decisions made by others.

Previously it was assumed that each network participant had limited ability to effect the marginal harmonic price. In the pressence of filters this might not be the case, due to the large affect filters have on the distortion magnitudes throughout the network, and the fact installed filter capacity may act as a disincentive towards others considering adding capacity. Simply put in a strong network, due to the physical realities any filter will potentially have a very large "market share" and hence influence on the price. (In the case of a weak system, harmonic externalities are by and large internalised and hence efficient behaviour should result in the absence of marginal pricing.) As such there may be an incentive for an individual to install an optimal amount of capacity, in the knowledge this will act as a disincentive to others building filters; but then inject a sub-optimal amount of filter current into the network so to maximise the value of those injections. However while a potential problem, in reality the costs of setting up a filter to act in such a way is likely to prove prohibitive. Also this problem ceases to exist in the absence of a harmonic market, but yet the marginal pricing tools can still be used by a network operator to establish the optimal level of filter investment.

Finally by way of example it was demonstrated in a marginal pricing environment, the choices made by loads with respect to reducing harmonic injections, and the installation of filters, will improve the welfare of each individual, and the network as a whole. It was shown harmonic pricing leads to large gains in welfare compared to the Toll Road method, on account of the fact the Toll Road method is unlikely to lead to efficient allocation of resources. Such methods developed with the objective of producing a fair allocation of costs, may in fact distribute the costs fairly, but it is unlikely these costs will equal the value provided by the filter. There in lies the key advantage of marginal pricing, it distributes an efficient level of costs among the loads fairly.

Chapter 6

PASSIVE FILTER INCLUSION

6.1 INTRODUCTION

In Chapter 5 it was that shown marginal pricing should produce efficient behaviour with respect to the installation of active filter capacity, if no installed capacity can be withheld from the market. In this chapter passive filters in a marginal pricing environment are investigated.

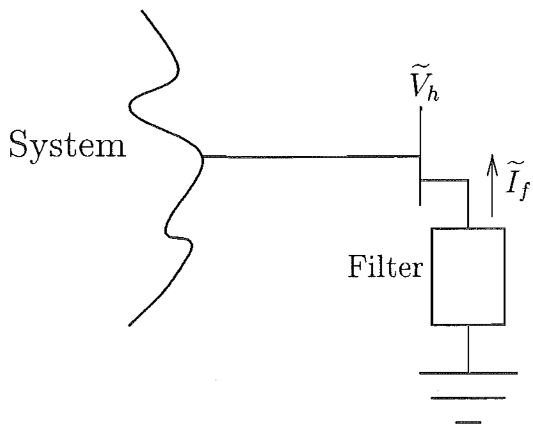


Figure 6.1 Passive harmonic filter model

For the purposes of this work, passive filters are characterised by their admittance:

$$\begin{aligned}\text{Admittance of filter installed at busbar } i &= \widetilde{C}_i \, pu \\ &= C_i e^{j\varsigma_i} \, pu\end{aligned}$$

It is assumed that the filters are rated in line with their admittance (i.e. as the admittance of the filter increases so does the current through the filter). The result being that the larger the admittance of the filter, the larger the required rating and the larger the expense. This allows the filter to be broadly characterised, without having to get into particulars, which only serve to confuse the general results presented here.

This representation of a filter does not conform to reality. In real life there are many more factors to assocaited with filter instillation. These other factors have been ignored so that only issues related to harmonic mitigation need be considered.

As in the the previous chapter, this work examines the general case where it is possible to install passive filter capacity at each busbar throughout the network. This is an unlikely out come in practice, and at times there are advantages to restricting the solution to a filter at a single busbar, as it greatly simplifies the results and conclusions drawn. Where any such restrictions are imposed they are explicitly stated.

The filter admittance matrix describes the amount of filter capacity connected to the network.

$$\text{Filter admittance Matrix} = [Y_f(\widetilde{\mathbf{C}})]$$
$$= \begin{pmatrix} \widetilde{C}_1 & 0 & \cdots & \\ 0 & \widetilde{C}_2 & & \\ \vdots & & \ddots & \\ & & & \widetilde{C}_n \end{pmatrix} \tag{6.1}$$

In this chapter the optimal allocation of passive filter resources is characterised. It is found the inclusion of a passive filter produces two potential candidates to act as the marginal price. The relative merits of these two potential marginal prices are investigated. To this point loads and filters have always been charged/payed based on their current injections. Some might think it natural that passive filter owners be rewarded on the basis of installed capacity. As such a set of marginal prices for installed filter capacity are developed. These prices are based on the marginal prices for harmonic current injections, and hence have similar properties. As was the case for active filters, the allocation of passive filter resources, is a dynamic optimisation problem. Yet modeling the problem as a static optimisation problem has many advantages, the consequences of misrepresenting the problem are detailed. Finally an example is constructed using the test system, to investigate the baviour of marginal prices of the presence of a passive filter, and to compare the efficiency of different marginal price formuations.

6.2 NETWORK OPTIMUM

The intial model of the problem uses a single period. As discussed in Chapter 5, this is not an accurate representation of the problem, but the results from this static model are essentially the same as those from the dynamic model. To find the optimal passive filter investment one needs to specify a cost function for the filters. As before, the passive filters are assumed to have an operating and a capital cost. The operating cost is assumed to be dependent on the actual amount of filter capacity connected to the network for the period. This cost might include the resulting degredation in component life, due to the current carried and voltage across the filter. It is thought that this operating cost will be close to zero and ignored without much loss in accuracy.

$$\text{Operating Cost of Passive Filter at Busbar } i = PV_i(C_i) \tag{6.2}$$

The main cost associated with any filter will be the capital cost. This depends on the rating/capacity of the filter installed, which is a function of the filter admittance. Note the model formulation does not preclude the possibility that more passive filter capacity may be installed than is actually connected to the system during the period. This is to keep the results as general as possible, not because it is a likely scenario.

$$\text{Cost of Passive Filter Capital For Filter at Busbar } i = PF_i(C_{imax}) \tag{6.3}$$

$$\therefore \text{Total Cost For Passive Filter at Busbar } i = PV_i(C_i) + PF_i(C_{imax}) \tag{6.4}$$

Where C_i = Connected filter admittance at busbar i

C_{imax} = Installed filter capacity at busbar i, and hence the maximum possible admittance at that busbar

Using the cost functions for the passive filter at each busbar (should they exist), allows the network filter cost functions to be specified.

$$\text{Total Operating Cost of Passive Filter Injections} = PV(\mathbf{C}) = \sum_i^n PV_i(C_i) \tag{6.5}$$

$$\text{Total Capital Cost of Passive Filter Injections} = PF(\mathbf{C_{max}}) = \sum_i^n PF_i(C_{imax}) \tag{6.6}$$

In this specification the ability of loads to reduce their injections into the network is ignored. This has no consequences for the main results presented here. If injection reductions are allowed, the result is the same as in Chapter 5, i.e. it is optimal for loads to reduce their injections into the system so long as the marginal cost of doing so is less the value the network places on those injections.

The single utility maximisation problem for the network as a whole is described by:

$$\text{Maximise} \quad U(\mathbf{V_h}) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) \tag{6.7}$$

$$\text{Subject to} \quad \widetilde{\mathbf{I}}_{\mathbf{h}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}} = \mathbf{0} \tag{6.8}$$

$$\mathbf{C} \leq \mathbf{C_{max}} \tag{6.9}$$

This problem can be solved using the Lagrangian in equation 6.10.

$$\begin{aligned}\mathcal{L} =& U(\mathbf{V_h}) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) \\ &+ \widetilde{\boldsymbol{\mu}}_h \left(\widetilde{\mathbf{I}}_{\mathbf{h}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}}\right) + \boldsymbol{\lambda}_{\boldsymbol{P}}(\mathbf{C_{max}} - \mathbf{C})\end{aligned} \tag{6.10}$$

The first order conditions of interest are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} - \widetilde{\boldsymbol{\mu}}_h \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}] = \mathbf{0} \tag{6.11}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = -\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \widetilde{\boldsymbol{\mu}}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} - \boldsymbol{\lambda}_{\boldsymbol{P}} = 0 \tag{6.12}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C_{max}}} = -\frac{\partial PF(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} + \boldsymbol{\lambda}_{\boldsymbol{P}} = \mathbf{0} \tag{6.13}$$

Equation 6.11, gives the new marginal prices where a passive filter is included in the network.

$$\widetilde{\boldsymbol{\mu}}_h = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}[e^{j\theta}]^{-1}\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \tag{6.14}$$

The inclusion of passive filter capacity changes the form of the marginal harmonic prices. The active filter only changes the marginal prices to the extent it alters the voltage. But the inclusion of a passive filter alters the responsiveness of the system to any harmonic injections, hence the marginal prices must change in structure to reflect this. This makes sense, as if with the inclusion of a passive filter the resultant voltage distortion for a given injection is changed, this implies the value of that injection to the network has changed, and hence the marginal price also should change.

Looking at equation 6.12, this can be rearranged to give:

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \lambda_P$$

$$\text{Marginal income for connected capacity} = \text{Marginal cost of connected capacity} + \text{Premium if filter capacity fully utilised} \tag{6.15}$$

Combining the above expression with equation 6.13, the optimal behaviour where the constraint is binding is given by:

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} \tag{6.16}$$

Using the result:

$$\text{Injection into the network from the passive filter } \widetilde{\mathbf{I}}_{\mathbf{f}} = -[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}} \tag{6.17}$$

Equation 6.16, can be interrupted as saying that the marginal income from installed filter capacity should equal the marginal cost of having it connected to the network, plus the marginal capital cost of filter capacity. As was the case with the active filter, the cost of capital term will be dominant, with the marginal operating/connection cost close to zero. This result is essentially identical to that of the active filter, the only difference being the inclusion of the passive filter changes the form of the harmonic prices.

6.2.1 Filter Admittance Angle

The first order conditions in equations 6.11 to 6.13 also implicitly define the optimal angle of the passive filter admittance. Looking at equation 6.16, both the two marginal cost terms ($\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}}$ and $\frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}}$) are real numbers. The implication is that:

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} = \text{A real number} \tag{6.18}$$

In establishing the above condition, of importance is the angle of harmonic voltage at each busbar ($\boldsymbol{\theta}$). It was demonstrated in Section 3.3 if there is no filter in the network, so that the branch admittances are much greater than the shunt admittances, the harmonic voltage angle is given by:

$$\theta_m = \beta^{-1} + \alpha \quad \forall\, m \tag{6.19}$$

If there is a shunt filter at busbar f, which has an admittance that is large compared to the other network admittances, the elements of the inverse admittance matrix can be approximated by:

$$\widetilde{y}_{ij}^{-1} = \frac{x_1 e^{jX_1}}{x_2 e^{jX_2}} \qquad \text{Where } i, j \neq f \tag{6.20}$$

$$\widetilde{y}_{ij}^{-1} = \frac{x_1 e^{jX_1}}{C_f e^{j\varsigma_f} + x_2 e^{jX_2}} \qquad \text{Where } i \text{ or } j = f \tag{6.21}$$

$$\text{Where } x_1, X_1, x_2, X_2 \text{ are all constants} \tag{6.22}$$

The expressions in equations 6.20 and 6.21, prove to be reasonably accurate where the included filter is large. It not possible to specify what is 'large', as that depends on circumstance. But in such cases, equation 6.19 still holds in a slightly modified form. All the elements of the inverse admittance matrix no longer have a constant phase angle, but those terms with a different phase angle have zero magnitude. As such:

$$\begin{aligned} \theta_m &= \beta_F^{-1} + \alpha \quad \forall\, m \\ \text{Where} \quad \beta_F^{-1} &= \text{Angle of elements of the inverse admittance} \\ &\quad\ \text{matrix that have non-zero magnitude} \\ &= X_1 - X_2 \end{aligned} \tag{6.23}$$

Substituting equation 6.23 into 6.14, produces the same result as in Chapter 5, the phase angle of the marginal price is equal to the negative angle of the injected harmonic current, plus π radians, if harmonic distortion is detrimental to individuals' utility. (Again in this section it has been assumed that all injections are made at a common angle).

$$\angle\widetilde{\mu}_h = -\alpha \qquad \text{Where } \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \geq 0 \tag{6.24}$$

$$\angle -\widetilde{\mu}_h = -\alpha \qquad \text{Where } \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} < 0 \tag{6.25}$$

Considering the case where harmonic distortion has negative impacts on each load's utility $(\frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} < 0)$, equation 6.16 suggests

$$\begin{aligned} [e^{-j\alpha}]\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}[e^{j\theta}] &= \begin{pmatrix} \left(e^{j(\varsigma_1+\beta_F^{-1})},0,...,0\right) & & \\ & \ddots & \\ & & \left(0,...,0,e^{j(\varsigma_n+\beta_F^{-1})}\right) \end{pmatrix} \\ &= \begin{pmatrix} (1,0,...,0) & & \\ & \ddots & \\ & & (0,...,0,1) \end{pmatrix} \end{aligned} \tag{6.26}$$

$$\Rightarrow \varsigma_i = -\beta_F^{-1}\ \forall\, \text{busbars } i, \text{ where there exists a passive filter} \tag{6.27}$$

Using the result in 6.27, the optimal passive filter current injections are found to have the same charactersitics as the optimal active filter injections.

$$\begin{aligned} &\text{Optimal passive filter injections } \widetilde{\mathbf{I}}_\mathbf{f} = -[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_\mathbf{h} \\ &\Rightarrow \widetilde{I}_{fi} = -C_i e^{j\varsigma_i} V_{hi} e^{j\theta_i} \\ &\quad = -C_i e^{-j\beta_F^{-1}} V_{hi} e^{j(\beta_F^{-1}+\alpha)} \\ &\quad = -C_i V_{hi} e^{j\alpha} \\ &\quad = C_i V_{hi} e^{j(\alpha+\pi)} \end{aligned} \tag{6.28}$$

The optimal filter injects harmonic current into the network π radians out of phase with the current injected by the distorting loads. As such marginal pricing produces the intuitive result that the optimal filter for the system will minimise the voltage distortion. In the unrealistic case where volatge distortion was viewed as positive by the loads, the opposite result would drop out, with equations 6.27 and 6.28 suggesting that the injected current from the filter should be in phase with that injected by the distorting loads. This makes sense in that if voltage distortion is of benefit for loads, the optimal filter current would be that which maximised the distortion. Were this is the case the optimal filter would need to have an impedance with a real negative component, and such would cease to be passive. Phasor diagrams showing the relative angles of network quantities are shown in Figures 6.2 and 6.3.

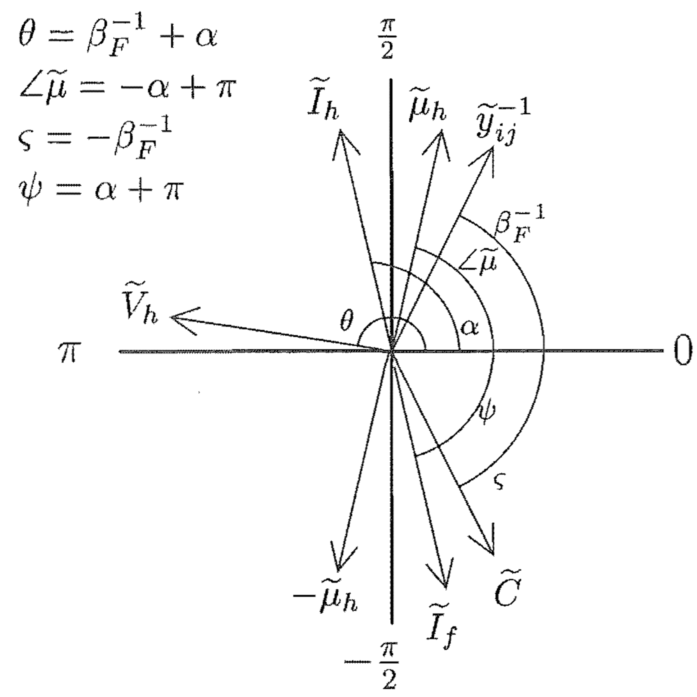


Figure 6.2 Phasor diagram of system quantities, where $\frac{\partial u_t(V_{ht})}{\partial V_{ht}} < 0 \; \forall t$

Fixed Filter Angle

The optimal admittance phase angle for any installed passive filter, is that which will minimise the prevailing voltage distortion through out the network. The flexibility to install a passive filter with any possible phase angle may not exist. Constraining this angle can only result in a reduction in welfare for the network. In examining the consequences of a fixed filter phase angle, the condition that defines the optimal phase angle must be revisited (equation 6.16):

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}}$$

This condition describes the optimal filter phase angle in that it requires:

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = \text{A real number}$$

These two conditions essentially state the marginal value of the installed capacity must equal its marginal cost, and that utility is a real number. But this is not the only interpretation of the conditions. An alternative interpretation is possible when generalising the concept of utility to a two dimensional framework, so that it becomes a complex number. In this environment specifing the optimal filter via the restriction that the utility from the marginal unit of installed filter capacity is a real number is to implicitly state any complex component of utility has no value. (To suggest that utility is multi-dimensional quantity is to stretch the generally accepted use of the concept, but where the different components are able to be manipulated to a single measure of value, one is back to the conventional concept of utility.) The optimal filter angle defined by equation 6.27 ($\varsigma_i = -\beta_F^{-1}$), can hence be interpreted as the condition which results in all utility from the marginal capacity being of value. If the passive filter angle is constrained

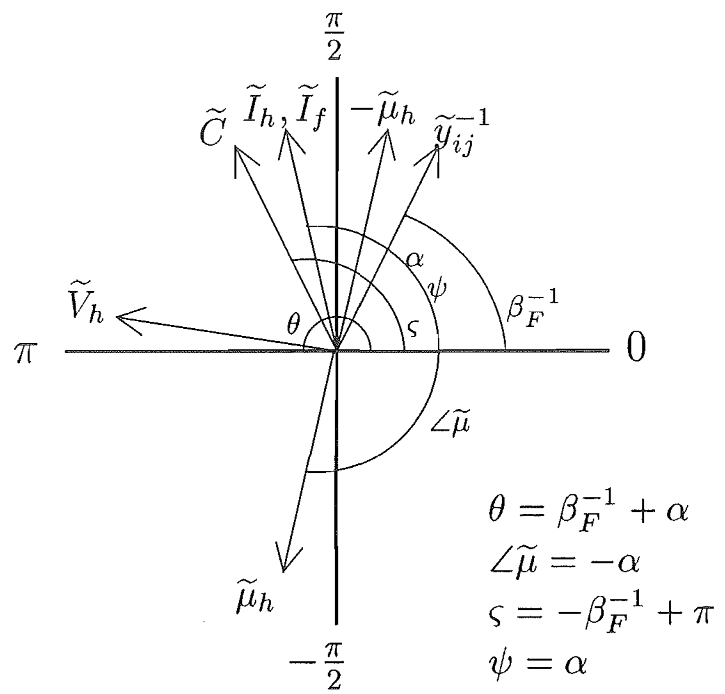


Figure 6.3 Phasor diagram of system quantities, where $\frac{\partial u_t(V_{ht})}{\partial V_{ht}} > 0\,\forall t$

to some value ς_i^*, the condition $\varsigma_i^* = -\beta_F^{-1}$, will not hold in general (unless one also constrains the solution to infinitely large passive filter at each busbar in the network, which obviously is not an efficient solution to the problem). In this environment of complex utility, the marginal utility of connected capacity will in general be a complex quantity.

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = x + jy \quad \text{(Some complex amount)} \tag{6.29}$$

If equation 6.18 describes the network optimum, only the real part of equation 6.29 has any value to the network (utility is a real entity), and the condition that describes the optimal amount of capacity is:

$$Re\{-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h}\} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} \tag{6.30}$$

In the case where the filter angle is constrained, without specifing the form of the passive filter cost functions, it is not possible to explicitly state what is the cost to the aggregate system of the constraint. The optimal passive filter capacity where the filter angle is constrained, is specified by equation 6.30. The optimal marginal prices for harmonic injections still have the same form as equation 6.14.

As a proxy for the cost to the network of constraining the filter angle, it is possible to use the marginal value to the network of a change in the filter angle.

$$\begin{aligned}\text{Marginal value of a change in } \varsigma &= Re\{-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \varsigma} \widetilde{\mathbf{V}}_\mathbf{h}\} \\ &= Re\{-j\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \widetilde{\mathbf{V}}_\mathbf{h}\}\end{aligned} \tag{6.31}$$

When the passive filter angle is variable and equation 6.27 is satified, the marginal value of a change in filter angle as defined above is zero. Where this value is non-zero, the welfare of the network could be improved if the phase of the filter admittance could be altered. Integrating the marginal value of a change in ς, from ς^* to $-\beta_F^{-1}$, indicates the cost to the network, resulting from the passive filter admittance being fixed.

$$\text{Cost to the network of a fixed filter angle} = \int_{-\beta_F^{-1}}^{\varsigma^*} Re\{-j\widetilde{\mu}_h[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_\mathbf{h}\}\partial\varsigma \tag{6.32}$$

6.2.2 Payment Imbalances

Before seeing if the harmonic prices described by equation 6.14, encourages efficient behaviour from network participants, the payments collected using such prices are detailed. It is assumed the harmonic property rights are allocated so that the loads have a right to a distortion free supply, hence:

$$\begin{aligned} \text{Payments due to loads as distortion compensation} &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}\mathbf{V_h} \\ \text{Payments made to passive filter} &= -\widetilde{\mu}_h[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_\mathbf{h} \end{aligned} \tag{6.33}$$

The amount collected from the nonlinear loads, for their harmonic injections is given by:

$$\begin{aligned} \text{Payments collected from harmonic injections} &= \widetilde{\mu}_h\widetilde{\mathbf{I}}_\mathbf{h} \\ &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}[e^{j\theta}]^{-1}\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_\mathbf{h} \\ &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}\mathbf{V_h} \end{aligned} \tag{6.34}$$

Equation 6.34 indicates the prices, described by equation 6.14, will collect enough to compensate the loads for voltage distortion seen, but will not cover the amount due to the passive filter. It is clearly a problem if the prices that are thought to represent the marginal value of injections fail to collect the correct amount. In the previous models this has never occurred.

One possible solution to the problem is to suggest that the optimisation problem for the aggregate network was structured incorrectly, and hence the marginal prices of equation 6.14, do not represent the true marginal value to the network of injected harmonics. An alternative formulation of the problem is to specify the filter cost as what the network pays for the filter injections.

$$\begin{aligned} \text{Alternative passive filter cost} &= \widetilde{\mu}_h^s\widetilde{\mathbf{I}}_\mathbf{f} \\ \widetilde{\mu}_h^s &= \text{Marginal price for harmonic current injections for this alternative formulation} \end{aligned} \tag{6.35}$$

Using this revised cost function the utility maximisation problem for the aggregate network is:

$$\text{Maximise} \quad U(\mathbf{V_h}) - \widetilde{\mu}_h^s\widetilde{\mathbf{I}}_\mathbf{f} \tag{6.36}$$

$$\text{Subject to} \quad \widetilde{\mathbf{I}}_\mathbf{h} - \left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_\mathbf{h} = \mathbf{0} \tag{6.37}$$

$$\mathbf{C} \leq \mathbf{C_{max}} \tag{6.38}$$

In this case the Lagrangian of the revised problem is:

$$\begin{aligned}\mathcal{L} &= U(\mathbf{V_h}) - \widetilde{\mu}_h^s \widetilde{\mathbf{I}}_\mathbf{f} + \widetilde{\mu}_h^s (\widetilde{\mathbf{I}}_\mathbf{h} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right) \widetilde{\mathbf{V}}_\mathbf{h}) + \lambda_P(\mathbf{C_{max}} - \mathbf{C}) \\ &= U(\mathbf{V_h}) + \widetilde{\mu}_h^s (\widetilde{\mathbf{I}}_\mathbf{h} - [Y_h]\widetilde{\mathbf{V}}_\mathbf{h}) + \lambda_P(\mathbf{C_{max}} - \mathbf{C})\end{aligned} \tag{6.39}$$

The first order condition of the Lagrangian is:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} - \widetilde{\mu}_h^s [Y_h][e^{j\theta}] = \mathbf{0} \tag{6.40}$$

The implied marginal prices resulting from this formulation of the problem are:

$$\widetilde{\mu}_h^s = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} [e^{j\theta}]^{-1} [Y_h]^{-1} \tag{6.41}$$

The alternative prices of equation 6.41, are identical to the prices in equation 6.14, were the passive filter excluded from the network. The amount collected for harmonic injections using these prices is given by 6.42.

$$\begin{aligned}\widetilde{\mu}_h^s \widetilde{I}_h &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} [e^{j\theta}]^{-1} [Y_h]^{-1} \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right) \widetilde{\mathbf{V}}_\mathbf{h} \\ &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} [e^{j\theta}]^{-1} [Y_h]^{-1} [Y_h] \widetilde{\mathbf{V}}_\mathbf{h} + \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} [e^{j\theta}]^{-1} [Y_h]^{-1} [Y_f(\widetilde{\mathbf{C}})] \widetilde{\mathbf{V}}_\mathbf{h} \\ &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \mathbf{V_h} - \widetilde{\mu}_h^s \widetilde{\mathbf{I}}_\mathbf{f}\end{aligned} \tag{6.42}$$

These alternative prices $\widetilde{\mu}_h^s$, collect exactly the correct amount to compensate loads and pay for the filter injections. This would tend to suggest that these prices are the optimal prices for the network. On the other hand the prices $\widetilde{\mu}_h^s$, are independent of the passive filter capacity, installed in the network. The prices $\widetilde{\mu}_h^s$ (equation 6.41) are identical to $\widetilde{\mu}_h$ (equation 6.14), in the case where there is no filter capacity present. Intuitively it would seem natural that the inclusion of a passive filter into the network should alter the value of harmonic injections, due to the fact it will alter the consequences of the injections. Therefore, using $\widetilde{\mu}_h^s$ will collect the correct amount, but will not provide the correct signals as to the value of each load's actions. The formulation of the problem that produces the prices $\widetilde{\mu}_h^s$, is clearly incorrect as $\widetilde{\mu}_h^s \widetilde{\mathbf{I}}_\mathbf{f}$, only represents payments between parties in the system, it is not the actual cost of the filter to the system as a whole. Hence using $\widetilde{\mu}_h^s \widetilde{\mathbf{I}}_\mathbf{f}$ to represent the cost of the filter is a misrepresentation, and can not lead to an optimal allocation of resources. A more detailed examination of the ability of both $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, to encourage efficient behaviour is covered next.

6.3 INDIVIDUAL INCENTIVES

The conditions that describe the optimal allocation of filter resources are detailed in equations 6.11 through to 6.13. But it is not entirely clear from these first order conditions, what are the correct marginal prices. Ideally the prices should encourage each load to act so the afore mentioned conditions are met, while at the same time collecting the correct amount from those making harmonic injections into the network (i.e. $\widetilde{\mu}_h^s \widetilde{I}_h = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \mathbf{V_h} - \widetilde{\mu}_h^s \widetilde{\mathbf{I}}_\mathbf{f}$).

As it turns out, neither set of prices $\widetilde{\mu}_h$ or $\widetilde{\mu}_h^s$, will encourage the optimal allocation of passive filter resources described by equation 6.16. First the behaviour that results from the prices $\widetilde{\mu}_h$ (equation 6.14), when a potential filter owner looks to maximise the profits they can extract from a series of filter installed across the network, will be considered. This approach is in contrast to the last chapter where the behaviour was described with respect to filter owner at some specific busbar i. The conclusions drawn are independent of how the problem is constructed. The problem can be modelled as:

$$\begin{aligned}
\text{Maximise} \quad & \widetilde{\mu}_h \widetilde{\boldsymbol{I}}_{\boldsymbol{f}} - PV(\mathbf{C}) - PF(\mathbf{C}_{\mathbf{max}}) = -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \widetilde{\boldsymbol{V}}_{\boldsymbol{h}} - PV(\mathbf{C}) - PF(\mathbf{C}_{\mathbf{max}}) && (6.43)\\
\text{Subject to} \quad & \mathbf{K} - \widetilde{\mu}_h \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}] = \mathbf{0} && (6.44)\\
& \widetilde{\mathbf{I}}_{\mathbf{h}} - \widetilde{\mathbf{I}}_{\mathbf{R}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}} = \mathbf{0} && (6.45)\\
& \mathbf{C} \leq \mathbf{C}_{\mathbf{max}} && (6.46)
\end{aligned}$$

Note that in equation 6.44, the loads' utility functions, with respect to voltage distortion have been assumed linear. Also note that it has been assumed the filter admittance angle is unconstrained. This problem has an associated Lagrangian:

$$\begin{aligned}
\mathcal{L} = & -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \widetilde{\boldsymbol{V}}_{\boldsymbol{h}} - PV(\mathbf{C}) - PF(\mathbf{C}_{\mathbf{max}}) + \left(\mathbf{K} - \widetilde{\mu}_h \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}]\right)\lambda_\mu \\
& + \lambda_v \left(\widetilde{\mathbf{I}}_{\mathbf{h}} - \widetilde{\mathbf{I}}_{\mathbf{R}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}}\right) + \lambda_C \left(\mathbf{C}_{\mathbf{max}} - \mathbf{C}\right)
\end{aligned} \tag{6.47}$$

Equation 6.47, has associated first order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{V}_{\mathbf{h}}} &= -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})][e^{j\theta}] - \lambda_v \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}] = \mathbf{0} && (6.48)\\
\frac{\partial \mathcal{L}}{\partial \widetilde{\mu}_h} &= -[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}]\lambda_\mu = \mathbf{0} && (6.49)\\
\frac{\partial \mathcal{L}}{\partial \mathbf{C}} &= -\widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathbf{h}} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}[e^{j\theta}]\lambda_\mu - \lambda_v \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathbf{h}} - \lambda_C = \mathbf{0} && (6.50)\\
\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\mathbf{max}}} &= -\frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} + \lambda_C = \mathbf{0} && (6.51)
\end{aligned}$$

Do the conditions described by equations 6.48 through to 6.51 match those that are optimal for the aggregate network (equations 6.11 through 6.13)? Starting with equation 6.48, this is easily rearranged to give:

$$\lambda_v = -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \tag{6.52}$$

There are two extreme values this Lagrange multiplier can take.

1. If very large passive filters exist throughout the network:

$$[Y_f(\widetilde{\mathbf{C}})] \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \approx [I] \Rightarrow \lambda_v = -\widetilde{\mu}_h \tag{6.53}$$

2. If there are no passive filters installed throughout the network:

$$[Y_f(\widetilde{\mathbf{C}})] = [0] \Rightarrow \lambda_v = \mathbf{0} \tag{6.54}$$

This result is expected as:

$$\lambda_v = \frac{\partial \text{Filter Owner Welfare}}{\partial \widetilde{\mathbf{I}}_{\mathbf{h}}} \tag{6.55}$$

If there is an infinite amount of filter capacity in the network, all injected harmonic current will be sunk by one of the filters, and hence the marginal value to the filter owner of the harmonic injections is the prevailing marginal price. But the value of the harmonic injections to the owner of a filter with zero admittance is zero.

Equation 6.49, is easily manipulated to yield the structure of λ_μ.

$$\lambda_\mu = -[e^{j\theta}]^{-1}\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}} \tag{6.56}$$

Again there are two extremes cases which yield the range of values λ_μ, can take.

1. Large passive filters throughout the network

$$\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}[Y_f(\widetilde{\mathbf{C}})] \approx [I] \Rightarrow \lambda_\mu = -\widetilde{\mathbf{V}}_{\mathbf{h}} \tag{6.57}$$

2. No passive filters installed in the network

$$[Y_f(\widetilde{\mathbf{C}})] = [\mathbf{0}] \Rightarrow \lambda_\mu = \mathbf{0} \tag{6.58}$$

The constraint values for the two extreme cases can be used to find the bounds on equation 6.50.

1. Where $[Y_f(\widetilde{\mathbf{C}})] = [\mathbf{0}]$, equation 6.50 reduces to

$$-\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathbf{h}} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \tag{6.59}$$

   This condition is identical to equation 6.16. This indicates that initially when there is no filter capacity in the system, the incentives at the margin are those required to produce optimal behaviour from a potential filter owner. That is, when there is no installed filter capacity in the network, the incentive exists for a potential owner to invest until the marginal benefit to the network equals the marginal cost of the filter.

2. Where $\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}[Y_f(\widetilde{\mathbf{C}})] \approx [I]$, equation 6.50 reduces to

$$\widetilde{\mu}_h \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathbf{h}} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \tag{6.60}$$

   The condition in equation 6.60, is different to the network optimum described in equation 6.16. It suggests the owner should reduce the amount of filter capacity in the system, or that investment in filter capacity should be made to the point where the marginal cost is equal to the marginal reduction in network utility as a result of the investment.

The second condition in equation 6.60 is somewhat hard to interpret, as it represents a nonsensical situation. But these two results can be used to bound the behaviour of a filter owner for the likely equilibrium situation where $\mathbf{0} \leq \mathbf{C} \leq \infty$. For such values of $\mathbf{C}$, the constraints will take on values

$$\begin{array}{l} -\widetilde{\mu}_h \leq \lambda_v \leq \mathbf{0} \\ \Rightarrow \lambda_v = -k_v \widetilde{\mu}_h \\ \text{Where } 0 \leq k_v \leq 1 \end{array} \tag{6.61}$$

$$
\begin{aligned}
&-\widetilde{\mathbf{V}}_{\mathrm{h}} \leq \lambda_\mu \leq \mathbf{0} \\
&\Rightarrow \lambda_\mu = -k_\mu \widetilde{\mathbf{V}}_{\mathrm{h}} \\
&\text{Where } 0 \leq k_\mu \leq 1
\end{aligned} \tag{6.62}
$$

Substituting this into equation 6.50 yields:

$$
\begin{aligned}
-\widetilde{\mu}_h(1-k_v-k_\mu)\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathrm{h}} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \\
\Rightarrow -\widetilde{\mu}_h\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathrm{h}} = \frac{1}{1-k_v-k_\mu}\left(\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \lambda_C\right) \\
\text{Marginal Income of Filter} \geq \text{Total Marginal Cost of Filter}
\end{aligned} \tag{6.63}
$$

Equation 6.63 describes the general behaviour of the filter owner. The filter owner will install filter capacity up to some point where the marginal income from filter capacity is in excess of the marginal cost. Or put another way, the marginal benefit to the network of extra installed capacity, exceeds the marginal cost of providing that capacity. Clearly this does not match the optimal condition for the network described in equation 6.16. This behaviour from the point of view of the filter owner can be understood in that as soon as a reasonably large filter is included in a strong network, it will sink a large proportion of the harmonic current. To increase the filter capacity serves to only reduce the value of all current sunk by the filter, for little gain in terms of increased filter current. This demonstrates a passive filter owner is unlikely to operate in an efficient manner, as a filter in a strong network inherently has a great deal of market power.

Having established that the prices $\widetilde{\mu}_h$, will not encourage efficient behaviour from filter owners, the other possible form of prices $\widetilde{\mu}_h^s$ (equation 6.41), are investigated next. As shown previously $\widetilde{\mu}_h^s$ collects the correct amount from the injectors of harmonic current, and hence if they were to prove as effective as $\widetilde{\mu}_h$, in encouraging efficient behaviour from filter owners, $\widetilde{\mu}_h^s$ would be the preferred form of harmonic prices. The corollary being that in calculating the optimal harmonic prices, no consideration of the amount of passive filter capacity in the network needs to be considered. The profit maximisation problem for a passive filter owner in this case is given by:

$$
\text{Maximise} \quad \widetilde{\mu}_h^s \widetilde{\boldsymbol{I}}_{\boldsymbol{f}} - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) = -\widetilde{\mu}_h^s[Y_f(\widetilde{\mathbf{C}})]\widetilde{\boldsymbol{V}}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) \tag{6.64}
$$

$$
\text{Subject to} \quad \widetilde{\mathbf{I}}_{\mathrm{h}} - \widetilde{\mathbf{I}}_{\mathrm{R}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathrm{h}} = \mathbf{0} \tag{6.65}
$$

$$
\mathbf{C} \leq \mathbf{C}_{\max} \tag{6.66}
$$

Note that there is no constraint in this case related to price, as $\widetilde{\mu}_h^s$ is independent of the level of installed filter capacity, and hence exogenous. The Lagrangian associated with this problem is given in 6.67.

$$
\begin{aligned}
\mathcal{L} = &-\widetilde{\mu}_h^s[Y_f(\widetilde{\mathbf{C}})]\widetilde{\boldsymbol{V}}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) + \lambda_{v*}\left(\widetilde{\mathbf{I}}_{\mathrm{h}} - \widetilde{\mathbf{I}}_{\mathrm{R}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathrm{h}}\right) \\
&+ \lambda_{C*}\left(\mathbf{C}_{\max} - \mathbf{C}\right)
\end{aligned} \tag{6.67}
$$

The first order conditions of interest are:

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{V}_{\mathrm{h}}} = -\widetilde{\mu}_h^s[Y_f(\widetilde{\mathbf{C}})][e^{j\theta}] - \lambda_{v*}\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}] = \mathbf{0} \tag{6.68}
$$

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = -\widetilde{\mu}_h^s\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathrm{h}} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_{v*}\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_{\mathrm{h}} - \lambda_{C*} = 0 \tag{6.69}
$$

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\max}} = -\frac{\partial PF(\mathbf{C}_{\max})}{\partial \mathbf{C}_{\max}} + \lambda_{C*} = \mathbf{0} \tag{6.70}
$$

In assessing how close the conditions in equations 6.68, through to 6.70 come to matching the system optimum (equations 6.11-6.13), the constraint term λ_{v*}, must be evaluated. Equation 6.68, is easily manipulated to yield

$$\lambda_{v*} = -\widetilde{\mu}_h^s[Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \tag{6.71}$$

Again there are two extreme cases, which will bound λ_{v*}.

1. Where there is no filter capacity installed in the system so that

$$[Y_f(\widetilde{\mathbf{C}})] = [0] \;\Rightarrow\; \lambda_{v*} = \mathbf{0} \tag{6.72}$$

2. Where there is an infinite amount of filter capacity in the network

$$\begin{gathered}[Y_f(\widetilde{\mathbf{C}})] \to [\infty] \text{ so that } [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \approx [I] \\ \Rightarrow \lambda_{v*} = -\widetilde{\mu}_h^s\end{gathered} \tag{6.73}$$

This result is identical to those previously, except that $\widetilde{\mu}_h^s$, replaces $\widetilde{\mu}_h$. The interpretation of the constraint term also is identical, as it represents the marginal value of harmonic injections to the filter owner. A filter owner with no installed capacity is indifferent to the amount of harmonic injections by the nonlinear loads. While if there is infinite capacity installed, so that all injections made by nonlinear loads are sunk by the installed filters, the marginal value of all such injections is $\widetilde{\mu}_h^s$.

In looking at equation 6.69, and substituting in equation 6.71 gives the result:

$$-\widetilde{\mu}_h^s\left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial\mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} - \frac{\partial PV(\mathbf{C})}{\partial\mathbf{C}} - \lambda_{C*} = \mathbf{0} \tag{6.74}$$

Of interest is to compare equation 6.74, with the network optimum given in equation 6.16. But before this can be done 6.74, must be restated in terms of $\widetilde{\mu}_h$. This is easily performed using the fact:

$$\widetilde{\mu}_h^s = \widetilde{\mu}_h\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} \tag{6.75}$$

As such equation 6.74, restated in terms of $\widetilde{\mu}_h$ is:

$$-\widetilde{\mu}_h\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1}\left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial\mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} - \frac{\partial PV(\mathbf{C})}{\partial\mathbf{C}} - \lambda_{C*} = \mathbf{0} \tag{6.76}$$

Again this expression is bounded by the two limiting cases of installed filter capacity.

1. Where there is no filter capacity in the network so $[Y_f(\widetilde{\mathbf{C}})] = [\mathbf{0}]$

$$-\widetilde{\mu}_h\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial\mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} - \frac{\partial PV(\mathbf{C})}{\partial\mathbf{C}} - \lambda_{C*} = \mathbf{0} \tag{6.77}$$

This equation is the network optimum given in equation 6.16. This indicates if there is no filter capacity installed in the network, potential filter owners have an incentive to act efficiently at the margin. Naturally both sets of prices $\widetilde{\mu}_h$, and $\widetilde{\mu}_h^s$ produce identical behaviour where there is no installed filter capacity, as if $[Y_f(\widetilde{\mathbf{C}})] = [\mathbf{0}]$, $\widetilde{\mu}_h = \widetilde{\mu}_h^s$.

2. Where there is an infinite amount of filter capacity installed so that $[Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1} = [I]$

$$-\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_{C*} = \mathbf{0} \tag{6.78}$$

Equation 6.78, states where there is an infinite amount of filter capacity installed the cost associated with installing filter capacity should be equal to zero.

It is possible to come up with a simplified representation of equation 6.76, for all the intermediate cases of $[\mathbf{0}] \leq [Y_f(\widetilde{\mathbf{C}})] \leq [\infty]$.

$$\begin{array}{c} -\widetilde{\mu}_h k_*(1-k_{**})\dfrac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} - \dfrac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_{C*} = \mathbf{0} \\ \text{Where} \quad 1 \leq k_* \leq \infty \\ 0 \leq k_{**} \leq 1 \end{array} \tag{6.79}$$

Using the results of the two extreme cases (equations 6.77 and 6.78), one can infer

$$\text{As } [Y_f(\widetilde{\mathbf{C}})] \to [\infty] \qquad k_*(1-k_{**}) \to 0$$

As such equation 6.79, can be simplified further to

$$\begin{array}{l} -\widetilde{\mu}_h k_{***}\dfrac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} - \dfrac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_{C*} = \mathbf{0} \\ \text{Where} \quad 0 \leq k_{***} \leq 1 \\ \qquad [Y_f(\widetilde{\mathbf{C}})] \to [\infty] \Rightarrow k_{***} \to 0 \\ \qquad [Y_f(\widetilde{\mathbf{C}})] \to [\mathbf{0}] \Rightarrow k_{***} \to 1 \end{array} \tag{6.80}$$

Equation 6.80 , is most easily interpreted with a slight rearrangement of terms.

$$-\widetilde{\mu}_h\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} = \frac{1}{k_{***}}\left(\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \lambda_{C*}\right) \tag{6.81}$$

Equation 6.81 indicates that the optimal behaviour for a filter owner, will result in payments for the filter injections which are in excess of the marginal filter costs. This differs from the system optimum, and therefore use of the prices, $\widetilde{\mu}_h^s$, will not result in a utility maximising amount of passive filter capacity connected to the network.

The next question is which set of prices, $\widetilde{\mu}_h$ or $\widetilde{\mu}_h^s$, will produce an amount of installed passive filter capacity that is closest to the optimal level given in equation 6.16. Consider the case where there will be only a single filter installed in the network. (This restriction is made as it tidies the result).

For the prices $\widetilde{\mu}_h$, the resulting condition describing behaviour (equation 6.50), can be manipulated (in this restricted case) to:

$$-\widetilde{\mu}_h\left([I] - 2[Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \lambda_C \tag{6.82}$$

While for the prices $\widetilde{\mu}_h^s$, the optimal behaviour of filter owners was shown to be described by:

$$-\widetilde{\mu}_h\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1}\left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}}\widetilde{\mathbf{V}}_\mathrm{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \lambda_{C*} \tag{6.83}$$

Both these conditions produce optimal behaviour where there is no filter capacity installed, but as filter capacity is installed, their behaviour drifts from optimality. To establish which is likely to produce behaviour closer to that of equation 6.16, there is a need to measure the relative rates at which the conditions in equations 6.82 and 6.83, drift from optimality as $[Y_f(\widetilde{\mathrm{C}})] \to [\infty]$.

$$\text{As } [Y_f(\widetilde{\mathrm{C}})] \to [\infty] \quad \begin{array}{l} [I] - 2[Y_f(\widetilde{\mathrm{C}})]\left([Y_h] + [Y_f(\widetilde{\mathrm{C}})]\right)^{-1} \to -[I] \\ \left([Y_h] + [Y_f(\widetilde{\mathrm{C}})]\right)[Y_h]^{-1}\left([I] - [Y_f(\widetilde{\mathrm{C}})]\left([Y_h] + [Y_f(\widetilde{\mathrm{C}})]\right)^{-1}\right) \to [0] \end{array}$$

Equation 6.82 drifts from optimal at a greater rate than 6.83. As such the prices $\widetilde{\mu}_h^s$, will encourage filter owners to connect an amount of filter capacity that is closer to optimal, than the prices $\widetilde{\mu}_h$. Therefore, as the prices $\widetilde{\mu}_h^s$, will collect the correct amount from harmonic polluters, and encourage more efficient behaviour, $\widetilde{\mu}_h^s$ would seem to be the preferred marginal harmonic pricing formulation.

6.3.1 Individual Market Power

The previous results assumed that the filter owner would choose to install some level of passive filter capacity $\mathbf{C}_{\mathbf{max}}$, and then some fraction of this capacity would be connected to the system so that $\mathbf{0} \leq \mathbf{C} \leq \mathbf{C}_{\mathbf{max}}$. It was also assumed that the filter owner had the ability to influence both the harmonic voltages, and the marginal harmonic prices throughout the network ($\widetilde{\mathbf{V}}_{\mathbf{h}}$ and $\widetilde{\mu}_h$). Under this scenario there is no incentive for the filter owner to connect an optimal amount of filter capacity to the network. This is because the way the problem has been constructed the filter owner was given complete market power, and allowed to behave as a monopolist. When the filter owner is no longer a price taker, they will install a less than optimal amount of capacity into the network, this reduces their costs and inflates the price.

An efficient passive filter allocation is unlikely using the prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, if the filter owners are in an entrenched monopoly position. But this does not mean the prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, will certainly fail to produce optimal behaviour. To know the likely outcome of using the different prices, it must be determined if an individual will be able to establish a monopoly position with respect to owning filter capacity and is similar to the situation that existed for the active filter. If an individual is allowed to install a large amount of filter capacity and then connect varying amounts to the network as they see fit, it will be possible to develop a monopoly position. This is because in a strong network, spare capacity makes any extra filter capacity connected to the network potentially uneconomic, as the incumbent possess the ability to drive the value of filter injections to zero. In this situation no second individual would rationally add filter capacity, and hence the incumbent would have monopoly power, and the ability to influence $\widetilde{\mu}_h$ and $\widetilde{\mathbf{V}}_{\mathbf{h}}$.

If the network is weak, the actions at any busbar will have limited consequence for harmonic distortion at other busbars, and therefore the harmonic prices will be increasingly decoupled depending on how weak the network is. Under such circumstances it will not be possible to develop substantial market power as spare filter capacity will not act a deterrent other than at the busbar where it is related. But in such a network all the problems from harmonic voltage distortion are largely internalised and therefore there is no need for harmonic prices to bring about efficient behaviour.

One potential way to stop this development of market power would be to impose the rule, that all installed filter capacity must be connected to the network. This may not be required, as in reality the owner of filter capacity may want to connect it, and leave it. It is questionable, if such any individual will have the inclination to switch passive filters in and out. Having said that, if all went to plan for the filter owner, they would own spare capacity that would never need to be touched. It would just sit there to deter others.

By imposing rules, which make the threat of entry into the market legitimate economically, the filter owners might act as if they were in a competitive market, even though there is a single individual with filter capacity. Look at the decision making process for the filter owner where they face the prices $\widetilde{\mu}_h$, and they have no market power. The problem can be described by:

$$\text{Maximise} \quad \widetilde{\mu}_h \widetilde{I}_f - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) = -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \widetilde{V}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) \quad (6.84)$$

$$\text{Subject to} \quad \mathbf{C} = \mathbf{C}_{\max} \quad (6.85)$$

Here there are no constraints related to the voltage magnitude or harmonic price, as the imposed rule that all available capacity must be connected makes the harmonic market competitive, and hence $\widetilde{\mu}_h$ and $\widetilde{\mathbf{V}}_\mathbf{h}$ are exogenous.

This constrained optimisation problem has an associated Lagrangian:

$$\mathcal{L} = -\widetilde{\mu}_h [Y_f(\widetilde{\mathbf{C}})] \widetilde{V}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) + \lambda_C (\mathbf{C}_{\max} - \mathbf{C}) \quad (6.86)$$

This has associated first order conditions

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = -\widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \quad (6.87)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\max}} = -\frac{\partial PF(\mathbf{C}_{\max})}{\partial \mathbf{C}_{\max}} + \lambda_C = \mathbf{0} \quad (6.88)$$

These two conditions are easily combined to produce

$$-\widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\max})}{\partial \mathbf{C}_{\max}} \quad (6.89)$$

This equation is identical to that which describes the optimal network conditions in equation 6.16. Therefore use of the prices $\widetilde{\mu}_h$, in an environment where filter owners do not posses market power (the ability to control $\widetilde{\mu}_h$ and $\widetilde{\mathbf{V}}_\mathbf{h}$), will result in an optimal allocation of filter resources.

On the other hand, when the prices $\widetilde{\mu}_h^s$ are used in an environment where no filter has any market power the problem for the filter owner can be summarised as:

$$\text{Maximise} \quad \widetilde{\mu}_h^s \widetilde{I}_f - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) = -\widetilde{\mu}_h^s [Y_f(\widetilde{\mathbf{C}})] \widetilde{V}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) \quad (6.90)$$

$$\text{Subject to} \quad \mathbf{C} = \mathbf{C}_{\max} \quad (6.91)$$

This problem has the associated Lagrangian

$$\mathcal{L} = -\widetilde{\mu}_h^s [Y_f(\widetilde{\mathbf{C}})] \widetilde{V}_h - PV(\mathbf{C}) - PF(\mathbf{C}_{\max}) + \lambda_{C*} (\mathbf{C}_{\max} - \mathbf{C}) \quad (6.92)$$

With the following first order conditions:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = -\widetilde{\mu}_h^s \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_{C*} = \mathbf{0} \quad (6.93)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\max}} = -\frac{\partial PF(\mathbf{C}_{\max})}{\partial \mathbf{C}_{\max}} + \lambda_{C*} = 0 \quad (6.94)$$

Again these two conditions are easily combined to produce:

$$-\widetilde{\mu}_h^s \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\max})}{\partial \mathbf{C}_{\max}} \quad (6.95)$$

This condition is identical to the network optimum given in equation 6.16, except $\widetilde{\mu}_h$ has been replaced by $\widetilde{\mu}^s_h$. Given that for all $[Y_f(\widetilde{\mathbf{C}})] > [0]$, $\mu^s_h > \mu_h$, equation 6.95, suggests that when the prices $\widetilde{\mu}^s_h$ are used, a larger than optimal amount of filter capacity will be installed in the network. This is not surprising as the addition of filter capacity reduces the value of harmonic injections (shown in $\widetilde{\mu}_h$). If the prices paid to filters for sinking harmonics fails to reflect this reduction in value, filter owners will continue to add capacity to the point where the marginal cost equals the marginal value of the injections, before any filter capacity was added.

Note that when the filter owners have no market power the prices given in equation 6.14 ($\widetilde{\mu}_h$), are preferred to the prices given in equation 6.41 ($\widetilde{\mu}^s_h$). Yet in the case where filter owners do have market power the prices $\widetilde{\mu}^s_h$, will come closer to encouraging efficient behaviour than $\widetilde{\mu}_h$. It seems odd that the prices that actually reflect the true value of harmonic injections to the system can be inferior in guiding the system towards an efficient allocation of resources. This happens because the prices $\widetilde{\mu}_h$ are dependent on the amount of connected filter capacity, where as $\widetilde{\mu}^s_h$ are not. Therefore the rewards to a filter owner from holding back capacity are much greater with $\widetilde{\mu}_h$, because not only are costs reduced, but the prices are increased.

Using the prices $\widetilde{\mu}_h$ may lead to an efficient equilibrium where a competitive market can be enforced, but the problem still exists in that, as shown in Section 6.2.2, $\widetilde{\mu}_h\widetilde{\mathbf{I}}_\mathrm{h}$, will fail to collect enough, to both compensate the loads for voltage distortion seen at their busbar, and to pay the passive filter for its injections into the network ($\widetilde{\mu}_h\widetilde{\mathbf{I}}_\mathrm{f}$). One obvious potential solution to this problem is to add the amount, $\widetilde{\mu}_h\widetilde{\mathbf{I}}_\mathrm{f}$, to the payments made by the distorting loads. This is essentially what is done when the prices $\widetilde{\mu}^s_h$, are calculated. As shown in Section 6.2.2, $\widetilde{\mu}^s_h\widetilde{\mathbf{I}}_\mathrm{h}$ will collect the correct amount, but will fail to provide incentives to filter owners. Moreover by having harmonic prices that fail to correctly signal the value of harmonic injections, loads will not make efficient decisions with respect reducing their harmonic injections into the network ($\widetilde{\mathbf{I}}_\mathbf{R}$). For example with a large passive filter in the network, the value of injections will probably fall to the point no resources should be committed by loads to reduce their injections into the network. But $\widetilde{\mu}^s_h$ will fail to reflect this change in harmonic value. As such the amount loads continue to pay for their injections ($\widetilde{\mu}^s_{hi}\widetilde{I}_{hi}$) may make it optimal for them to allocate resources towards reducing their injections, where the amount spent exceeds the value the aggregate network places on the reduction.

If $\widetilde{\mu}_h$ is charged on the basis that it is desirable to achieve an efficient allocation of resources, the payments to the filter owners will need to come from "out of the system" (the harmonic market, not the actual physical system). One possible solution could be to incorporate the payments to the filter owner ($\widetilde{\mu}_h\widetilde{\mathbf{I}}_\mathrm{f}$), into the consumers' line charges (based on fundamental energy consumption). By separating the filter payments from the harmonic economic system, an efficient allocation of harmonic resources will result, and everything will be paid for. The disadvantage to shifting this cost burden elsewhere is that another market (and hence resource allocation) will become distorted. Loads that do not inject any harmonic current into the system could quite justifiably feel aggrieved about their fundamental line charges being inflated to pay for a filter, on which they place no burden. These inflated line charges, may result in them altering their behaviour with respect to fundamental power consumption on the basis of harmonic injections from other loads. This is not an efficient situation.

It would seem there is no simple way around this problem, in that the prices that encourage efficient behaviour fail to collect the correct amount of money. The fact this problem exists, might seem strange as to this point the marginal prices have been faultless in they have both encouraged efficient behaviour, and the books have balanced, even in the presence of active filters. The problem is the prices calculated, represent the value of a pure harmonic injection into the network.

$$\widetilde{\mu}_h = \frac{\partial \text{System Utility}}{\partial \widetilde{\mathbf{I}}_\mathbf{h}} \tag{6.96}$$

The harmonic injection from a nonlinear load into the network ($\widetilde{I}_{hi}$), is different to the current through a passive filter, as the filter current is associated with a change the network admittance matrix. Given the two currents are of a different nature, attempting to charge for filter injections as if they were not associated with a change in the network will have problems. No problem existed for the active filter as it was modelled as a current source, the same model used for the nonlinear loads. As modelled active filter injections are not associated with changes in the network admittance matrix, and therefore the marginal prices are efficient, and collect the correct amount. This revenue reconciliation problem for passive filters is further detailed in Chapter 7.

6.3.2 Alternative Formulation

It was demonstrated if filter owners are paid for the injections of the filter, it may be in their best interests to build capacity, which is not connected to the system, but acts as a disincentive to others thinking of adding capacity to the network. One solution to this problem is to pay passive filter owners an amount based on the filter capacity they have installed in the network, instead of the current through the filter. The optimal price to pay for filter capacity, will reflect what that capacity is worth to the network. This amount is closely linked to the marginal price for injections calculated previously.

Looking at the optimisation problem that faces the aggregate network, this is unchanged from previously (Section 6.2), except that all available capacity will be connected.

$$\begin{aligned}
\text{Maximise} \quad & U(\mathbf{V_h}) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) && (6.97)\\
\text{Subject to} \quad & \widetilde{\mathbf{I}}_\mathbf{h} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_\mathbf{h} = \mathbf{0} && (6.98)\\
& \mathbf{C} = \mathbf{C_{max}} && (6.99)
\end{aligned}$$

The first order conditions associated with this problem are given in equations 6.11 to 6.13. What is required is a marginal price for connected capacity, which encourages filter owners to act so that the optimal conditions are meet. A natural candidate is to pay filter owners an amount that corresponds to the marginal value of filter capacity. That is, pay the filter owners a price $\widetilde{\mu}_C$ given by:

$$\begin{aligned}
\widetilde{\mu}_C &= \frac{\partial \text{ System Utility}}{\partial \mathbf{C_{max}}}\\
&= -\widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}}_\mathbf{max})]}{\partial \mathbf{C_{max}}} \widetilde{\mathbf{V}}_\mathbf{h}
\end{aligned} \tag{6.100}$$

$\widetilde{\mu}_h$, is calculated the same way as it was previously (equation 6.14). With the price $\widetilde{\mu}_C$ for installed capacity a potential filter owner faces the following optimisation problem:

$$\begin{aligned}
\text{Maximise} \quad & \widetilde{\mu}_C \mathbf{C_{max}} - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) && (6.101)\\
\text{Subject to} \quad & \mathbf{C} = \mathbf{C_{max}} && (6.102)
\end{aligned}$$

The associated Lagrangian and first order conditions are detailed below

$$\mathcal{L} = \widetilde{\mu}_C \mathbf{C_{max}} - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) + \lambda_C(\mathbf{C_{max}} - \mathbf{C}) \tag{6.103}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = -\frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \tag{6.104}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\mathbf{max}}} = \widetilde{\mu}_C - \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} + \lambda_C = \mathbf{0} \tag{6.105}$$

Equations 6.104 and 6.105 are easily combined to produce:

$$\widetilde{\mu}_C = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} \tag{6.106}$$

Equation 6.106, combined with 6.100 produce a condition identical to the optimal condition shown in Section 6.2. It was implied here that the filter owner had little ability to influence the price of filter capacity, this condition should be met as by paying owners an amount based on the total installed capacity, there is nothing to be gained from not connecting capacity to the network. Connected or not, they receive an amount based on the assumption all available capacity is connected.

The fact that prices for filter capacity can be developed that encourage efficient behaviour should come as no great surprise, as the price for filter capacity was only a tweaking of the price for injected currents, and hence it is expected that all the previous results will hold true for this formulation. In fact it is impossible to calculate the optimal price for capacity without calculating the optimal price for injections. This formulation of the problem though has value in that for some it may seem more natural to pay filter owners for their installed capacity, instead of current through the filter. Though each approach is equivalent to each other, sometimes semantics count, and the capacity payments may be more attractive to some and as mentioned it may be easier to ensure an efficient outcome. Note that even when the prices for capacity are used to pay the filters, the prices $\widetilde{\mu}_h$, will still need to be used to charge distorting loads, and these charges will fail to collect the amount $\widetilde{\mu}_C\mathbf{C}_{\mathbf{max}}$, required to pay the filter. The same revenue reconciliation problems that exist for $\widetilde{\mu}_h$ will also exist for $\widetilde{\mu}_C$.

6.3.3 Optimal filter angle

The filter owner must also decide what is the prefered phase angle of the passive filter admittance, along with what amount of capacity to install. Using the same logic as previously, it can be suggested that the optimal phase angle for the filter angle is implicitly defined in equation 6.89. This condition is identical to that of the network optimum and as such must have the same implications. Specifically the phase of the filter admittance is equal to the negative values of the elements of network admittance.

$$\varsigma = -\beta_F^{-1}$$

But, as discussed in Chapter 4, the filter owner will only infact be paid the real part of $\widetilde{\mu}_h\widetilde{\boldsymbol{I}}_f$. This helps characterise how the filter owner may behave if the phase angle of the filter admittance is constrained. Rewritting the profit maximisation for the filter owner to more accurately reflect the problem they will in fact face, gives:

$$\begin{aligned} \text{Maximise} \qquad & Re\{\widetilde{\mu}_h\widetilde{\boldsymbol{I}}_f\} - PV(\mathbf{C}) - PF(\mathbf{C}_{\mathbf{max}}) && (6.107)\\ & = Re\{-\widetilde{\mu}_h[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}}\} - PV(\mathbf{C}) - PF(\mathbf{C}_{\mathbf{max}}) \\ \text{Subject to} \qquad & \mathbf{C} = \mathbf{C}_{\mathbf{max}} && (6.108) \end{aligned}$$

This problem demonstrates that concept of the loads only paying the real part of $\widetilde{\mu}_h\widetilde{\boldsymbol{I}}_f$ is essentially the same as the condition used to characterise the network optimum, i.e. the loads only attribute value to the real part of the utility function. The associated Lagranian of problem is:

$$\mathcal{L} = \sum_{i}^{n} \mu_{hi} C_i V_{hi} \cos\left(\varsigma_i + \beta_F^{-1}\right) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) + \lambda_C \left(\mathbf{C_{max}} - \mathbf{C}\right) \tag{6.109}$$

The the first order conditions for this problem are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = \left(\mu_{h1} C_1 V_{h1} \cos\left(\varsigma_1 + \beta_F^{-1}\right), \dots, \mu_{hn} C_n V_{hn} \cos\left(\varsigma_n + \beta_F^{-1}\right)\right) - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \lambda_C = \mathbf{0} \tag{6.110}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C_{max}}} = -\frac{\partial PF(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} + \lambda_C = \mathbf{0} \tag{6.111}$$

$$\frac{\partial \mathcal{L}}{\partial \varsigma} = \left(-\mu_{h1} C_1 V_{h1} \sin\left(\varsigma_1 + \beta_F^{-1}\right), \dots, -\mu_{hn} C_n V_{hn} \sin\left(\varsigma_n + \beta_F^{-1}\right)\right) = \mathbf{0} \tag{6.112}$$

Obviously equation 6.112 has multiple solutions. Looking at the second derivative:

$$\begin{array}{c} \frac{\partial^2 \mathcal{L}}{\partial \varsigma^2} = \left(-\mu_{h1} C_1 V_{h1} \cos\left(\varsigma_1 + \beta_F^{-1}\right), \dots, -\mu_{hn} C_n V_{hn} \cos\left(\varsigma_n + \beta_F^{-1}\right)\right) < \mathbf{0} \\ \text{Where } \varsigma_i = -\beta_F^{-1} \, \forall \, i \end{array} \tag{6.113}$$

These conditions in equations 6.110 to 6.113 produce the same result inferred earlier, given the ability to choose the admittance of the passive filter, it is optimal for the filter owner to choose a phase angle $\varsigma_i = -\beta_F^{-1} \, \forall \, i$. Also in the case where the filter phase angle is constrained the filter owner faces an optimisation condition identical to that which characterised the network optimum:

$$Re\{-\widetilde{\mu}_h \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}}\} - \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} - \frac{\partial PF(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} = \mathbf{0}$$

6.4 DYNAMIC PASSIVE FILTER OPTIMISATION

Modelling the passive filter optimisation problem as a single period problem is not an accurate representation of reality. The decision as to allocation of passive filter resources will cover an extended period of time, in which there will be many different prices for harmonic injections. As was the case for the active filter, modelling the decision making process as a multi-period problem does not change any of the major results, or reveal any new insights. The main difference being the interpretation of one Lagrange multiplier differs slightly. Due to the close similiarity, and in the interests of brevity, the previous passive filter problems are not detailed in a multi-period framework. However the basic network optimisation conditions, and the basic behaviour of loads when in a competitive environment are shown in a multi-period frame work for completeness. The notation used here to differentiate between busbar and temporal variation is identical to the notation used in Chapter 5.

6.4.1 Network Optimum In Dynamic Environment

Here the ability of loads to reduce their injections is not included. This exclusion has no consequences for explaining for the main results of this example. The optimisation problem for the network as a whole is:

$$\begin{aligned}
\text{Maximise} \quad & U(\underline{\mathbf{V_h}}) - PVD(\underline{\mathbf{C}}) - PFD(\underline{\mathbf{C_{max}}}) && (6.114)\\
\text{Subject to} \quad & \underline{\widetilde{\mathbf{I}}_\mathbf{h}} - \left([[Y_h]] + [[Y_f(\underline{\widetilde{\mathbf{C}}})]]\right)\underline{\widetilde{\mathbf{V}}_\mathbf{h}} = \underline{\mathbf{0}} && (6.115)\\
& \underline{\mathbf{C}} \leq \mathbf{C_{max}}\underline{[I]} && (6.116)
\end{aligned}$$

Where the passive filter cost functions in this dynamic envirnoment are:
Total Operating Cost of Passive Filter Injections = $PVD(\underline{\mathbf{C}})$
Total Capital Cost of Passive Filter Injections = $PFD(\underline{\mathbf{C_{max}}})$

The associated Lagrangian is:

$$\begin{aligned}
\mathcal{L} =& U(\underline{\mathbf{V_h}}) - PVD(\underline{\mathbf{C}}) - PFD(\underline{\mathbf{C_{max}}}) + \underline{\widetilde{\mu}_h}\left(\underline{\widetilde{\mathbf{I}}_\mathbf{h}} - \left([[Y_h]] + [[Y_f(\underline{\widetilde{\mathbf{C}}})]]\right)\underline{\widetilde{\mathbf{V}}_\mathbf{h}}\right)\\
&+ \underline{\lambda_P}\left(\mathbf{C_{max}}\underline{[I]} - \underline{\mathbf{C}}\right)
\end{aligned} \quad (6.117)$$

The first order conditions of interest are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \underline{\mathbf{V_h}}} &= \frac{\partial U(\underline{\mathbf{V_h}})}{\partial \underline{\mathbf{V_h}}} - \underline{\widetilde{\mu}_h}\left([[Y_h]] + [[Y_f(\underline{\widetilde{\mathbf{C}}})]]\right)[[e^{j\underline{\theta}}]] = \underline{\mathbf{0}} && (6.118)\\
\frac{\partial \mathcal{L}}{\partial \underline{\mathbf{C}}} &= -\frac{\partial PVD(\underline{\mathbf{C}})}{\partial \underline{\mathbf{C}}} - \underline{\widetilde{\mu}_h}\frac{\partial [[Y_f(\underline{\widetilde{\mathbf{C}}})]]}{\partial \underline{\mathbf{C}}}\underline{\widetilde{\mathbf{V}}_\mathbf{h}} - \underline{\lambda_P} = \underline{\mathbf{0}} && (6.119)\\
\frac{\partial \mathcal{L}}{\partial \mathbf{C_{max}}} &= -\frac{\partial PFD(\underline{\mathbf{C_{max}}})}{\partial \mathbf{C_{max}}} + \underline{\lambda_P}\underline{[I]} = \underline{\mathbf{0}} && (6.120)
\end{aligned}$$

Equation 6.118 to 6.120 state the conditions which must be meet each time period. The only difference being that the capital costs are able to be recovered over multiple periods. Therefore the excess in marginal income per unit of filter capacity over the marginal operating costs, must over the life of the filter sum to the marginal cost of filter capital (equation 6.121). Or simply stated, the capital need only pay for itself over an extended period versus a single period previously.

$$\sum_i^T \lambda_{Pi} = \frac{\partial PFD(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} \quad (6.121)$$

6.4.2 Filter Owner Optimum In Dynamic Environment

The behaviour of the filter owner is characterised in a dynamic environment. The problem a potential filter owner faces is given by:

$$\begin{aligned}
\text{Maximise} \quad & E[\underline{\widetilde{\mu}_h \widetilde{\mathbf{I}}_\mathbf{f}}] - PVD(\underline{\mathbf{C}}) - PFD(\underline{\mathbf{C_{max}}}) && (6.122)\\
\text{Subject to} \quad & \underline{\mathbf{C}} \leq \mathbf{C_{max}}\underline{[I]} && (6.123)
\end{aligned}$$

This can be solved using the Lagrangian, i.e.

$$\mathcal{L} = -E[\underline{\widetilde{\mu}_h}][[Y_f(\underline{\widetilde{\mathbf{C}}})]]\underline{\mathbf{V_h}} - PVD(\underline{\mathbf{C}}) - PFD(\underline{\mathbf{C_{max}}}) + \underline{\lambda_C}\left(\mathbf{C_{max}}\underline{[I]} - \underline{\mathbf{C}}\right) \quad (6.124)$$

The associated first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \underline{\mathbf{C}}} = -E[\underline{\widetilde{\mu}_h}]\frac{\partial[[Y_f(\underline{\widetilde{\mathbf{C}}})]]}{\partial \underline{\mathbf{C}}}\underline{\widetilde{\mathbf{V}}_\mathbf{h}} - \frac{\partial PVD(\underline{\mathbf{C}})}{\partial \underline{\mathbf{C}}} - \underline{\lambda_C} = \underline{0} \tag{6.125}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{\mathbf{max}}} = -\frac{\partial PFD(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} + \underline{\lambda_C[I]} = \underline{0} \tag{6.126}$$

These conditions are identical to those in equations 6.118 through 6.120, except the filter owner decisions are based on expectations of the future price. The implications of this are identical to active filter case detailed in Chapter 5, and hence will not be repeated.

Modelling the decision making process with respect to passive filters, as a static optimisation problem is an obvious misrepresentation. But this section demonstrates that extending the model to incorporate its dynamic nature serves to only complicate the notation, as the results and conclusions are essentially unchanged. The dynamic model basically states that the results from the static optimisation model must be met each time period.

6.5 PASSIVE FILTER EXAMPLE

Using the test system detailed in Appendix A, the behaviour of marginal prices in the presence of a passive filter are investigated. For the purposes of this example, there is only a single filter installed at busbar one. As the test system is strong, the consequences of moving the filter are limited.

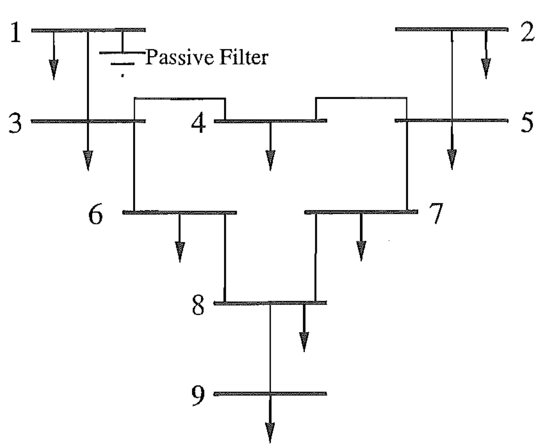


Figure 6.4 Test system with included passive filter

6.5.1 Marginal Pricing With Filter Variation

There are two potential formulations for the marginal price ($\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$), both of which will be affected the magnitude and phase of the filter admittance (though in the case of $\widetilde{\mu}_h^s$, these effects are only second order effects). One way in which the inclusion of the passive filter will affect the marginal prices is via its affect on the harmonic voltages throughout the network. Figure 6.5 displays the resulting harmonic voltage magnitude at busbar three for a different range of filter admittances.

Figure 6.5 shows that that in general the addition of filter capacity (increasing the admittance of the filter), will reduce the magnitude of the harmonic voltage at busbar three, this is as expected. Figure 6.6 shows the variation in the harmonic voltage phase angle at busbar three for a range of possible filter admittances. For a given filter admittance magnitude, the voltage

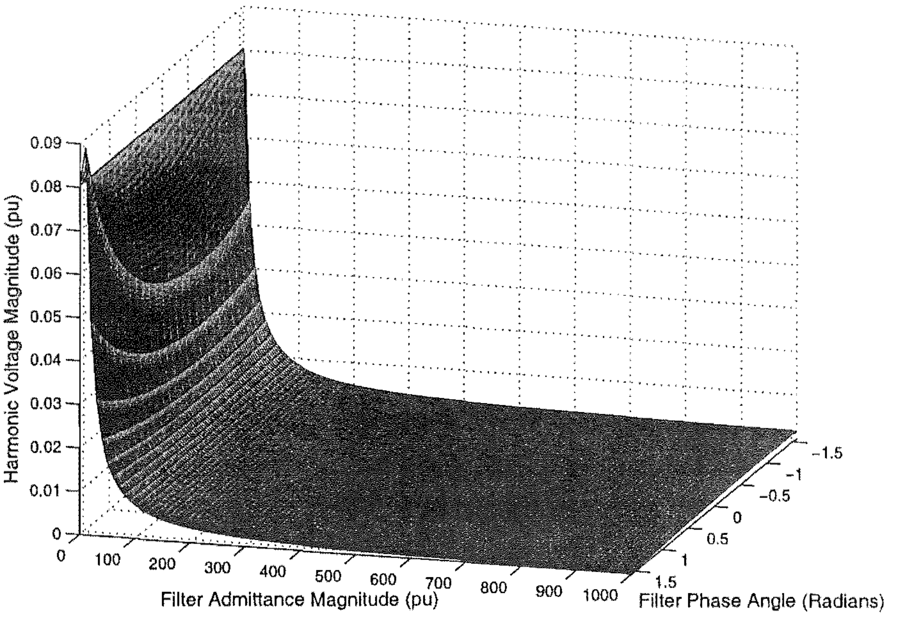


Figure 6.5 Resultant harmonic voltage magnitude at busbar three with inclusion of passive filter

phase at busbar three approximates a monotonically decreasing function of the filter phase angle. Equivalent diagrams for each busbar in the network are contained in Appendix D.

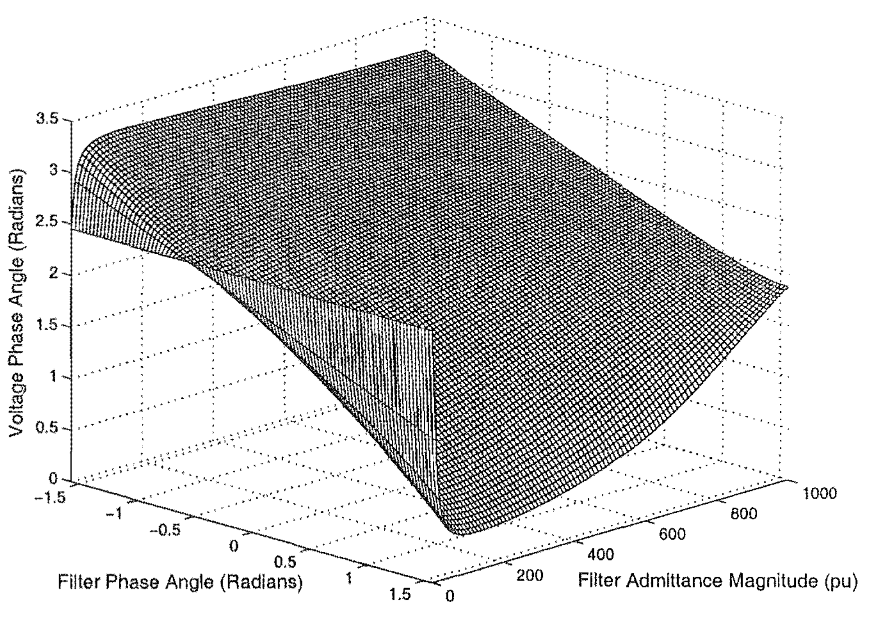


Figure 6.6 Resultant harmonic voltage phase angle at busbar three with inclusion of passive filter

Equation 6.27 describes the optimal passive filter phase angle. It was suggested this phase angle would minimise the harmonic voltage distortion throughout the network. Figure 6.7 shows the voltage distortion throughout the network as as the phase angle of the passive filter (ς) is varied (for three different sized filters). For the test system of Appendix A, $\beta_F^{-1} \approx 0.61$. Figure 6.7 demonstrates that a phase angle $\varsigma = -\beta_F^{-1}$ does indeed minimise the voltage throughout the network.

Figure 6.7 indicates that where the admittance of the installed filter is large, the phase angle of the filter has little influence on the harmonic voltage magnitude. Figure 6.8 shows the harmonic voltage magnitude and angle for each of the nine busbars where there is an installed

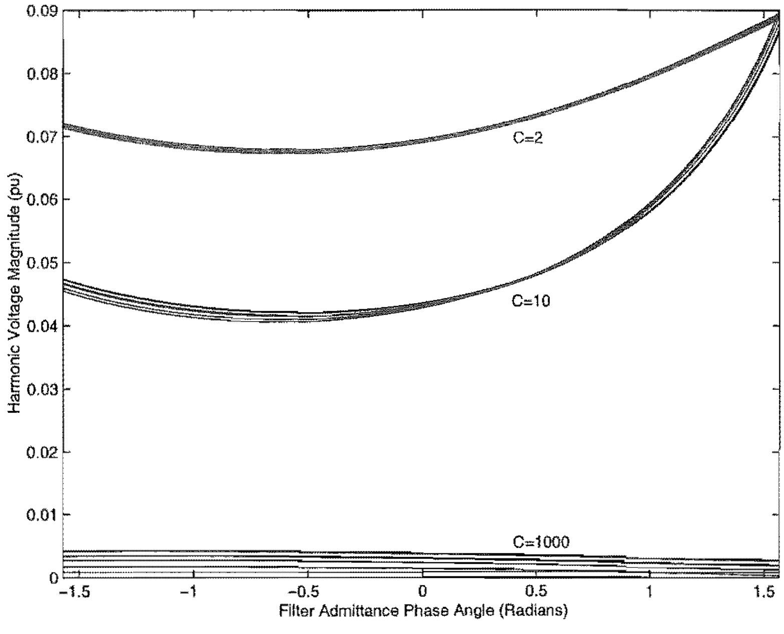
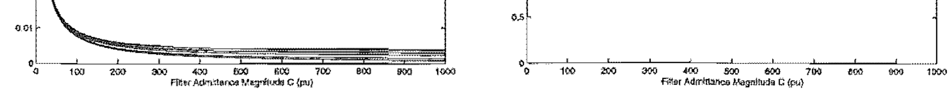


Figure 6.7 Resultant harmonic voltage throughout the network as the passive filter phase angle is varied. Three different magnitudes of filter admittance are considered.

filter of variable admittance magnitude, with an optimal admittance phase angle ($\varsigma = -\beta_F^{-1}$). In Appendix D similar figures are shown for filters that look capacitive, resistive and inductive.



(a) Harmonic Voltage Magnitude $\mathbf{V_h}$ (b) Harmonic Voltage Phase θ

Figure 6.8 Variation in the harmonic voltage at each busbar as the magnitude of the passive filter a busbar one is varied. The phase angle of the filter is the optimal angle from equation 6.27

There were two different forms of marginal harmonic prices put forward. In one case the harmonic prices were calculated after the inclusion of the filter $\widetilde{\mu}_h$, while the presence of the filter was excluded in calculating the prices $\widetilde{\mu}_h^s$. Both sets of prices will be influenced by the inclusion of the filter to the extent that the harmonic voltage throughout the network is influenced.

Figure 6.9 shows the variation in the magnitude of $\widetilde{\mu}_{h3}$ as the characteristics of the filter at busbar one are altered. As the admittance of the filter increases, the magnitude of the harmonic price decreases. The response of the harmonic price magnitude to changes in the filter admittance is identical to the response of the voltage magnitude shown in Figure 6.5. This is as expected given that the value of the injections into the network are dependent of the resultant harmonic voltage. Figure 6.9, also shows that as the admittance of the filter increases in magnitude, the influence of the phase angle on the harmonic price magnitude quickly diminishes towards zero.

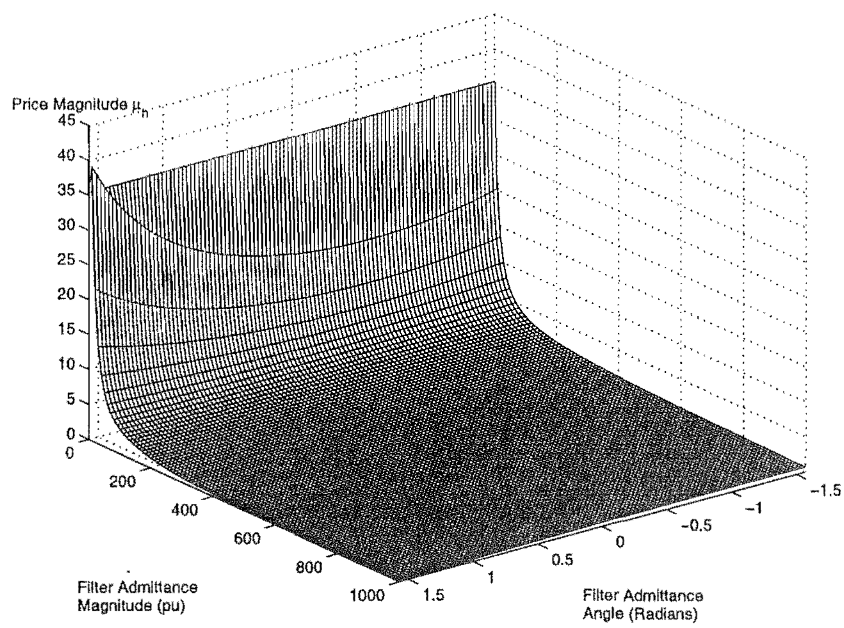


Figure 6.9 Variation in the magnitude of the harmonic price at busbar three (μ_{h3}), as a result in variation in the passive filter admittance

Again this reflects the behaviour of the harmonic voltage as the filter increases in size.

The angle of the harmonic price at busbar three is shown to be fairly stable in Figure 6.10, except in the case where the angle of the filter is highly capacitive, and hence the magnitude of the included filter does have a significant influence on the phase of the marginal prices.

Figure 6.11 displays the variation in magnitude of the harmonic price $\widetilde{\mu}^s_{h3}$, as the filter admittance is varied. As the harmonic prices $\widetilde{\mu}^s_h$ do not explicitly account for the included filter, the influence of the filter magnitude on the price magnitude is expected to be minimal. This is shown to be the case in Figure 6.11. A consequence of this is that these prices are unable to encourage efficient filter investment. Figure 6.12 shows that the phase angle of the prices $\widetilde{\mu}^s_h$, will be heavily influenced by the included filter.

Figures 6.13 and 6.14, show the magnitude and phase angle of $\widetilde{\mu}_h$ and $\widetilde{\mu}^s_h$ as the magnitude of the passive filter at busbar one is varied between 0 and 100 pu. In these figures the phase angle of the filter admittance is the optimal value $\varsigma = -\beta_F^{-1}$. As expected the magnitude of the prices, $\widetilde{\mu}_h$, decrease rapidly as the filter admittance increases, while the prices $\widetilde{\mu}^s_h$, are essentially unaffected. Where the included filter has a phase angle $\varsigma = -\beta_F^{-1}$, the phase of the harmonic voltage will be largely unaffected by the inclusion of the filter and hence the prices should to be largely unaffected. But this property is specific to load utility functions that are linear in the voltage distortion seen at their busbar. If the utility function of the loads were a nonlinear function of the voltage distortion, $\widetilde{\mu}^s_h$ would be influenced by the magnitude of the included filter. Looking at the angle of the prices, in the case of $\widetilde{\mu}^s_h$, these are relatively constant and the angle at each busbar is equal. With $\widetilde{\mu}_h$, the angles are shown to increasingly diverge as the voltage and filter admittance magnitude fall towards zero.

Having seen how the voltage and marginal prices behave for different installed passive filters, the following graphs show the variations in harmonic payments made to/by the loads, as the filter admittance changes. In all cases these payments are based on the assumption that loads have a right to a clean supply and hence charge for distortion seen at their busbar. Figure 6.15 shows how the harmonic utility of each loads change as the filter capacity is increased. The phase angle of the harmonic filter admittance is the optimal angle. Naturally the utility of each load and the network mirrors the behaviour of the harmonic voltage magnitude. Under marginal clean

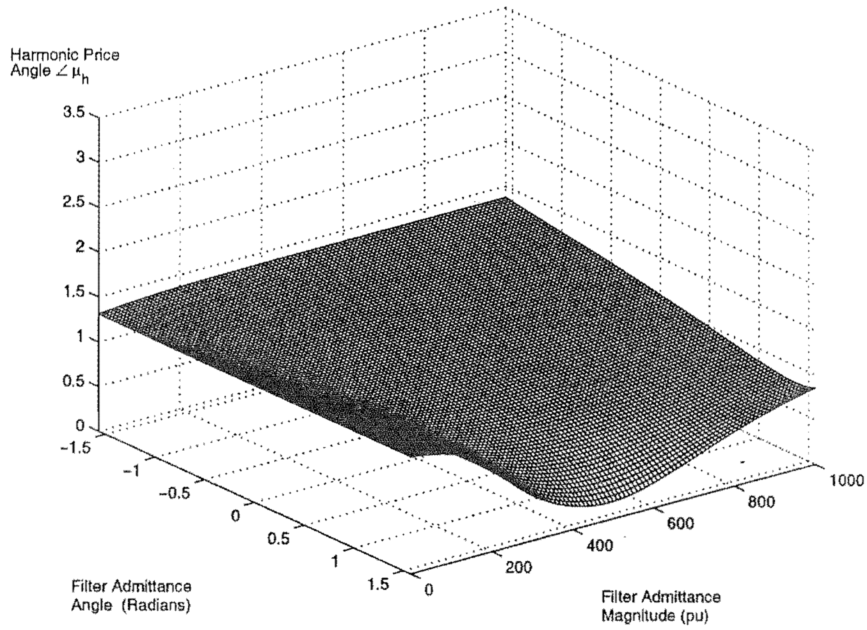


Figure 6.10 Variation in the angle of the harmonic price at busbar three ($\angle\widetilde{\mu}_{h3}$), as a result in variation in the passive filter admittance

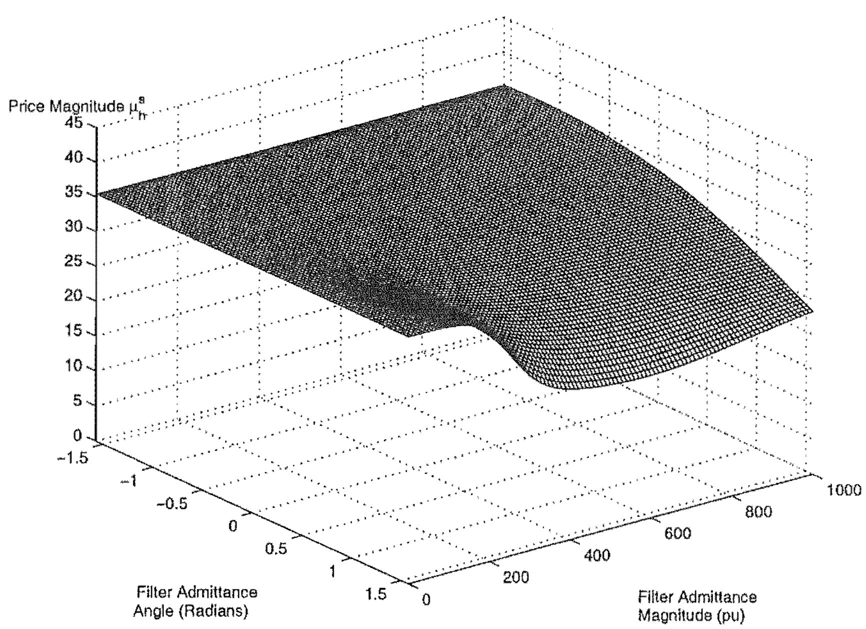


Figure 6.11 Variation in the magnitude of the harmonic price at busbar three (μ^{s}_{h3}), as a result in variation in the passive filter admittance

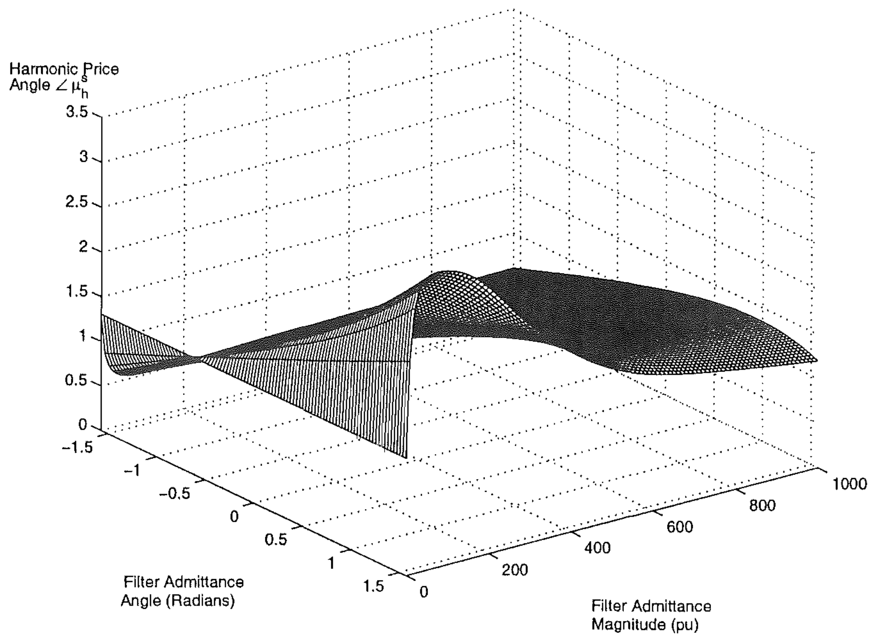
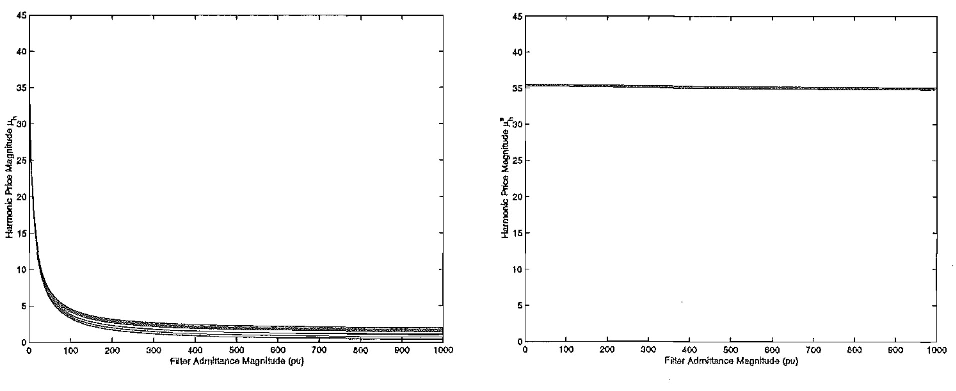
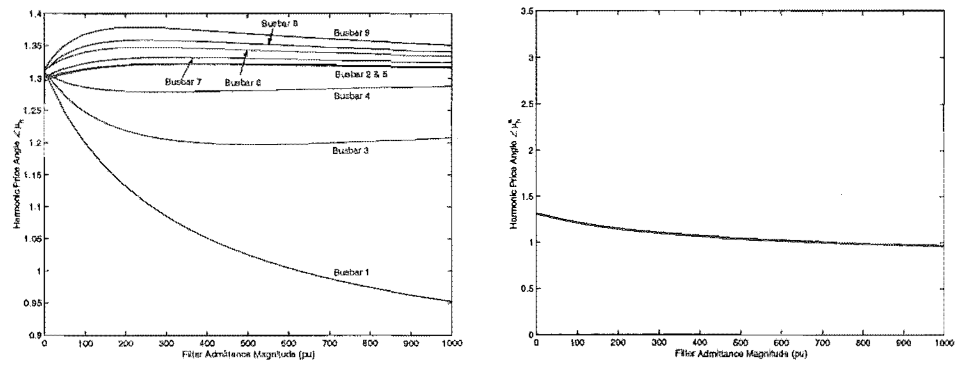


Figure 6.12 Variation in the angle of the harmonic price at busbar three ($\angle\widetilde{\mu}_{h3}^{s}$), as a result in variation in the passive filter admittance



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

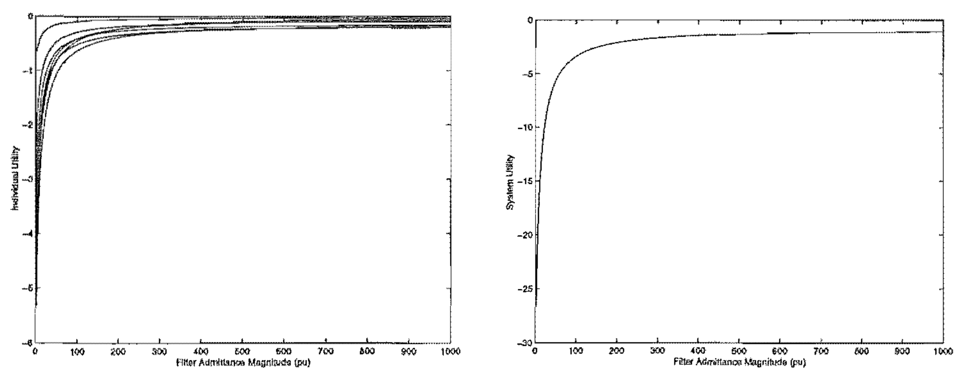
Figure 6.13 Variation in the magnitude of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of the optimal filter at busbar one is varied



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure 6.14 Variation in the angle of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of the filter at busbar one is varied

pricing the compensation each load receives for the distortion at their busbar is the negative value of their individual utility. Therefore as the filter acts to reduce the distortion throughout the network, the utility of each load increases, and as such they receive less compensation from the distorting loads.

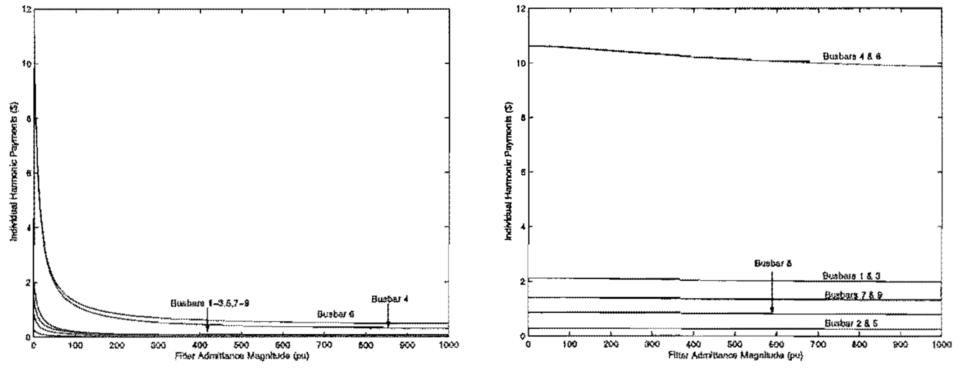


(a) Individual Utility (b) Total Network

Figure 6.15 Variation in the individual load and total system utility as the magnitude of the passive filter admittance is varied from 0pu to 1000pu

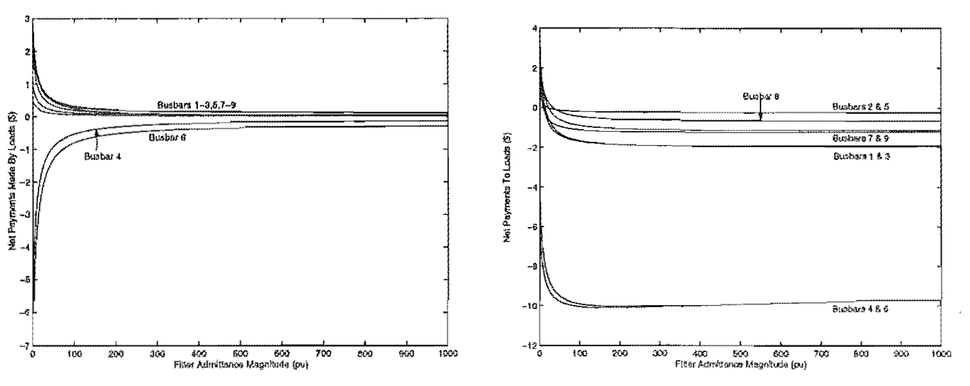
The harmonic payments made by each of the loads as the magnitude of the optimal filter is increased are shown in Figure 6.16. When the prices $\widetilde{\mu}_h$ are used, the payments made by the loads quickly decrease with increasing filter capacity. This is expected as the reduction in harmonic voltage means the value of the loads' injections must be reduced. On the other hand with the prices $\widetilde{\mu}_h^s$ the payment due from each load is largely unaffected by the size of the installed filter. Therefore any load, which injects a large proportion of the harmonic current into the network will prefer the prices $\widetilde{\mu}_h$. This is shown in Figure 6.17, which shows the net harmonic payments received by each load as the filter admittance is varied in magnitude. When the net harmonic payment for load i, is the compensation received due to distortion ($u_i(V_{hi})$ Figure 6.15) less the amount paid for harmonic injections ($Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\}/Re\{\widetilde{\mu}_{hi}^s\widetilde{I}_{hi}\}$ Figure 6.16).

All loads are better if $\widetilde{\mu}_h$ is used, compared with the prices $\widetilde{\mu}_h^s$. The reason for this being the revenue reconcilliation problem detailed in Section 6.2.2. If the prices $\widetilde{\mu}_h$ are used no revenue is recovered from the distorting loads to pay the filter for the injected filter current. While the



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

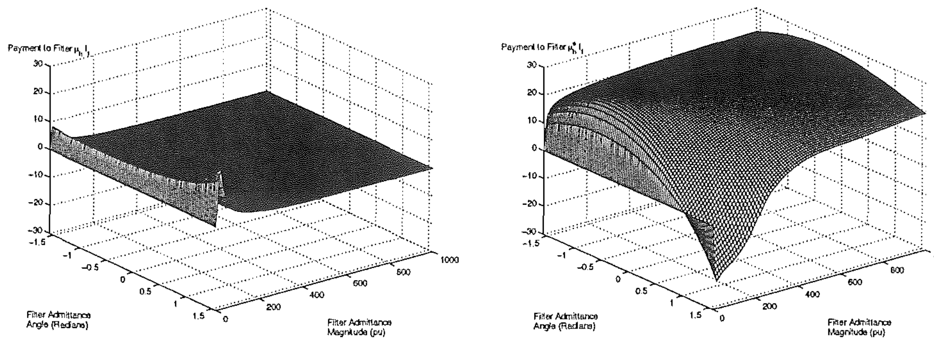
Figure 6.16 Harmonic payments made by the distorting loads under the two different pricing systems, as the magnitude of the passive filter admittance is varied from 0pu to 100pu



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure 6.17 Net harmonic payments made to the loads under the two different pricing systems, as the magnitude of the passive filter admittance is varied from 0pu to 100pu prices $\widetilde{\mu}_h^s$, do ensure enough is collected from the harmonic injections to compensate loads for the voltage distortion at their busbar and to pay the filter for its injections.

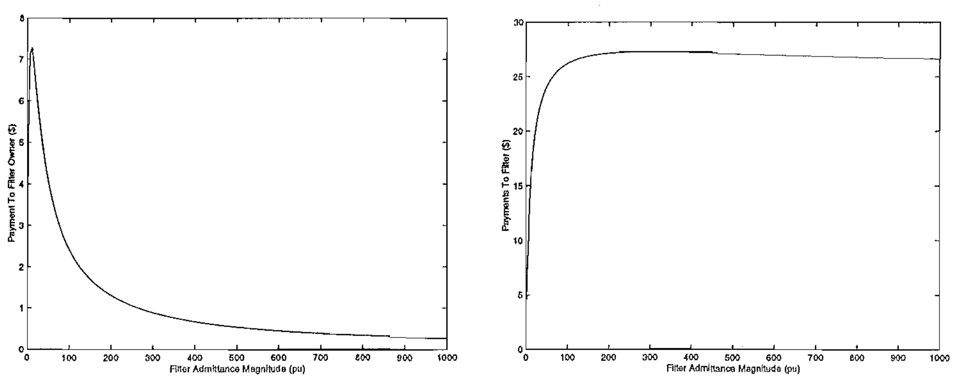
The payments for the filter current injected into the network will be heavily dependent on the admittance of the filter installed at busbar one. The payments to the filter, as a function of the filter admittance are shown in Figure 6.18 for both forms of marginal pricing.



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure 6.18 Payments to the filter under the two different sets of marginal prices, as a function of the filter admittance

The payments to the filter owner are much larger using the marginal prices $\widetilde{\mu}_h^s$. With these prices the marginal income from each unit of filter current is always positive. For the prices $\widetilde{\mu}_h$, it can be seen that the marginal income from the addition of filter capacity quickly turns negative. Hence for any given filter cost function, it can be seen that the prices $\widetilde{\mu}_h^s$, will encourage much more capacity to be installed than $\widetilde{\mu}_h$. Variation in filter payment as the admittance magnitude is varied ($\varsigma_i = -\beta_F^{-1}$), is shown in Figure 6.19.



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure 6.19 Payments to the filter under the two different sets of marginal prices, as a function of the filter admittance

Figure 6.19 illustrates the value of a monopoly position to the owner of the filter, if the prices $\widetilde{\mu}_h$ are used. Variation in the size of the installed filter capacity, has a huge effect on the amount of revenue collected and there is a clear level of installed capacity that will maximise the payments to the filter. Unless steps are taken to ensure the threat of added capacity is plausable, the owner of the filter capacity will naturally operate at this point (cost permitting), irrespective of what is the optimal amount of capacity for the network. With the prices $\widetilde{\mu}_h^s$, the filter owner has less market power (ability to influence the harmonic price and hence revenues collected), as such the amount a capacity added under $\widetilde{\mu}_h^s$, will primarily be dependent on the cost of adding filter capacity. As such using the prices $\widetilde{\mu}_h^s$, results in behaviour that mimics a competitive market, except the prices while largely independent, produce the wrong signals.

6.5.2 Relative Efficiency of Marginal Price Formulations

Next the outcomes from using the different marginal prices are compared. In developing this example the test system detailed in Appendix A was used. In each case where there is a filter included in the network it is at busbar one. In this example the expected outcomes under four different scenarios are detailed:

1. No filter included in the network, the prices $\widetilde{\mu}_h$ are used
2. A filter installed at busbar one, where the prices $\widetilde{\mu}_h$ are used
3. A filter installed at busbar one, where the prices $\widetilde{\mu}_h^s$ are used
4. A filter installed at busbar one, where the prices $\widetilde{\mu}_h$ are used, but the filter admittance is constrained to being capacitive

In each case the cost function associated with the installation of a filter was of the form:

$$\text{Cost of Passive Filter} = 0.001C^2 \tag{6.127}$$

The numbers in the cost function are choosen for illustration purposes. This precludes excludes loads from reducing their injections at a given cost.

In Figures 6.20 and 6.21 the harmonic voltage and price magnitude at each of the busbars are shown. In the final three scenarios, the filter admittance was choosen to maximise the payments to the filter. The filter was assumed to have no market power, hence the included filters are those which would result from a competitive market under each of the different pricing systems. The magnitude of the filter for each of the different cases is shown in Table 6.1. These figures demonstrate what a large effect the inclusion of a passive filter has on the value of injected harmonics, and hence the marginal harmonic price (scenario 1 compared with scenario 2). In equilibrium the inclusion of the filter reduces the value of all harmonic injections to about a sixth their original value. This ability to quantify the value/cost of harmonic injections and included filters is necessary if resources are to be allocated efficiently. Where the prices $\widetilde{\mu}_h^s$, are used the harmonic marginal prices hardly respond to the inclusion of the filter. The result being it is optimal for the filter owner to install a filter with an admittance over twice as large as the optimal filter when the prices $\widetilde{\mu}_h$, are used. But the cost function of equation 6.127, and voltages displayed in Figure 6.20, show despite a three fold increase in the marginal cost of filter capacity, the marginal reduction in voltage magnitude in scenario 3 compared with scenario 2 is small. This demonstrates the reduction in efficiency that results from a pricing system that fails to signal true marginal costs.

Table 6.1 Magnitude of included passive filter

	Scenario 2	Scenario 3	Scenario 4
Passive Filter Magnitude (pu)	47.9	115.4	56.5

Constraining the phase angle of the included filter has few consequences for the optimal filter magnitude and the resultant price magnitude. If the example had been constructed so that filter capacity was more expensive and the optimal filters had a smaller admittance, this

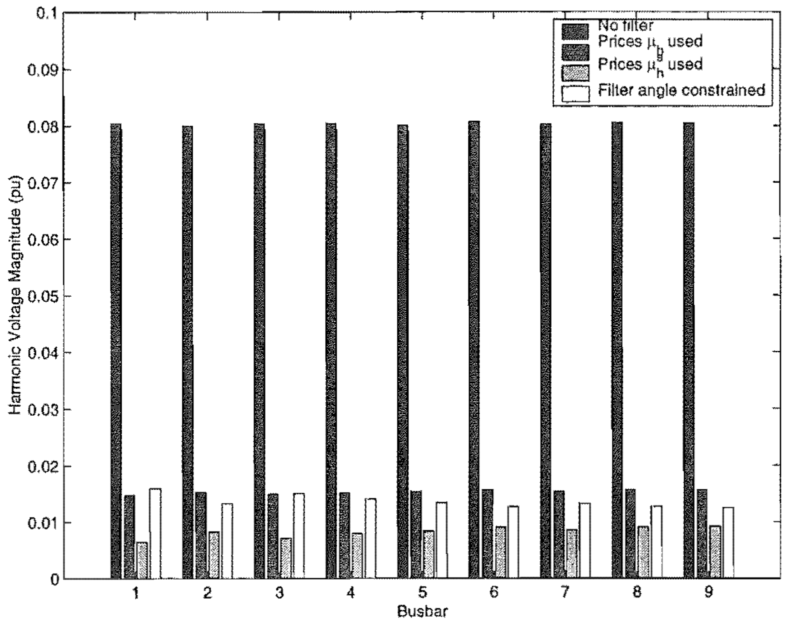


Figure 6.20 Harmonic voltage magnitude at each busbar, under the different scenarios

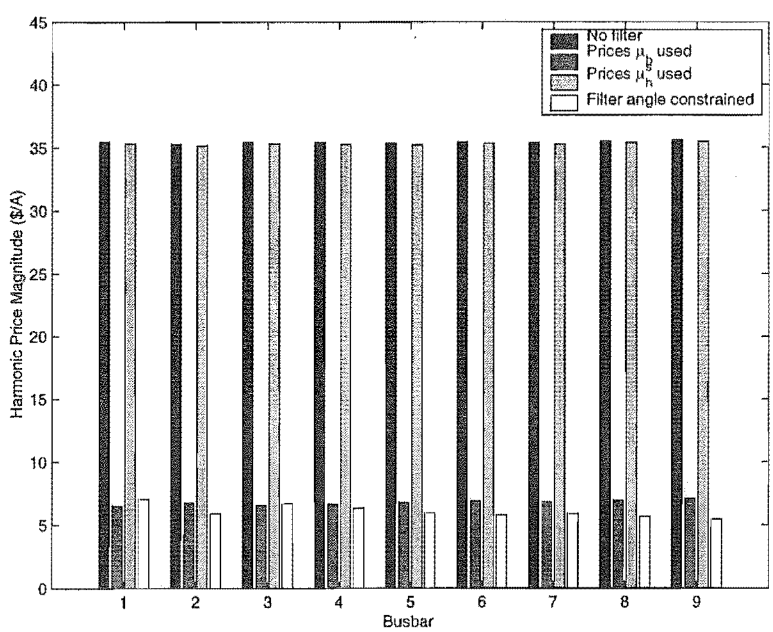


Figure 6.21 Magnitude of the harmonic marginal prices at each busbar under the different scenarios

constraint would of had more influence. Once the included filter is large the influence of its phase angle is limited.

The net payments made by each load is the compensation each load receives for distortion seen at their busbar, less the harmonic payments each load makes for their current injections. In this example it has been assumed each load has a right to a clean voltage supply. Where a filter is included and prices $\widetilde{\mu}_h$ are used, each load's net payments are maximised. This is a product of the revenue reconciliation problem detailed earlier.

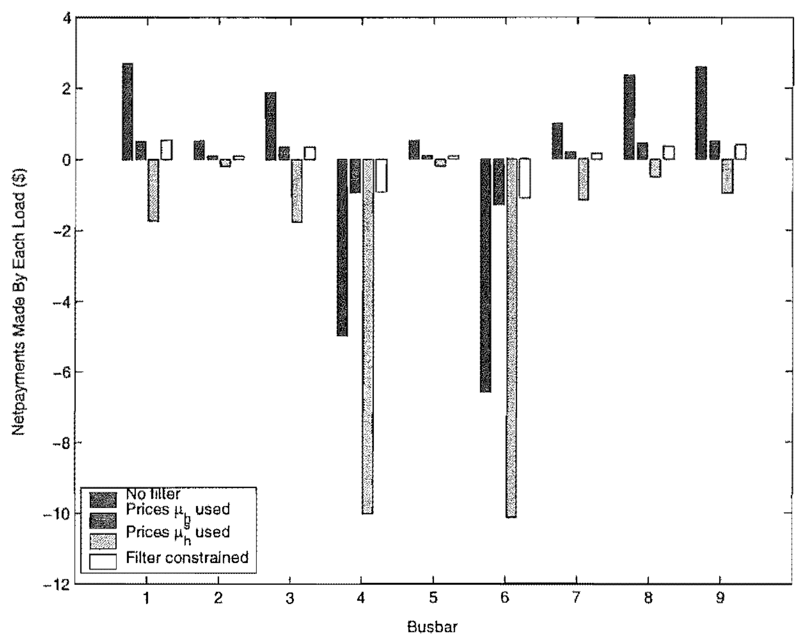


Figure 6.22 Net harmonic payments received by each load $(Re\{\widetilde{\mu}_{hi}\widetilde{I}_{hi}\} - u_i(V_{hi}))$ under the different scenarios

Table 6.2 details the filter revenue and costs, along with the total system utility for each scenario. The first thing to note is the increase in filter profits when the prices $\widetilde{\mu}_h^s$ are used. The increased profit to the filter owner is associated with a large decrease in efficiency, compared to the case where the prices better reflect the true value of injections $(\widetilde{\mu}_h)$. In this example the ability of loads to reduce their injections at some cost was ignored. Had it been considered, the reduction in efficiency associated with using $\widetilde{\mu}_h^s$ would have been greater, as not only would the prices $\widetilde{\mu}_h^s$ encourage non-economic filter capacity, they would also encourage loads to make non-economic reductions in their injections. As shown, the inclusion of a filter quickly reduces the value of harmonic injections, if loads fail to see this fall in value, they cannot be expected to behave efficiently. In all cases the inclusion of the filter was found to improve network welfare. It may seem odd that the scenario with the constrained filter angle resulted in the filter owner making more profit then when the filter angle was unconstrained. The filter admittance was constrained to phase angle that resulted in a small voltage resonance at the local busbar, which increased the value of the filters injections. In a competitive market this enhanced price at busbar one would be capitalised on by other filter owners, adding appropriately phased capacity and reducing the voltage to a level comparable with scenario 2. In this simple example with a single filter, this could not happen. Though this does serve to demonstrate the potential consequences of allowing filter owners market power, specifically it will be in their interest to install filters that instead of minimising the local voltage, manipulate the local price so the value of the injected filter currents are maximised.

Table 6.2 Filter Revenue and System Utility

Scenario	Filter Revenue	Filter Cost	Filter Profit	System Utility
1	0.00	0.00	0.00	−29.76
2	4.59	2.29	2.30	−7.96
3	26.63	13.32	13.31	−16.33
4	6.39	3.19	3.20	−8.33

6.5.3 Toll Road Pricing Comparison

The allocation of passive filter costs is something Toll Road pricing was designed for. It is difficult to make a comparison of the relative merits of marginal pricing and Toll Road pricing as it is impossible to know what filter would be installed were Toll Road pricing used. As demonstrated, failure to install an efficient amount of capacity has a large effect on total system welfare. Toll Road pricing fails to provide any signals as to what the admittance of the included filter should be. Under Toll Road pricing the filter owner is guaranteed to receive revenue in line with the cost of the filter, hence the incentive exists to install excess filter capacity.

In this example each load injected harmonics into a strong network at a common angle. Hence if the same filter were to be included in both cases, the distribution of harmonic charges under marginal and Toll Road pricing would look very similar. As soon as some loads started injecting at a phase angle greater than $\pi/2$ radians out of phase, marginal pricing and the Toll Road method will result in very different distributions of the harmonic costs. This is as marginal pricing recognises injections that reduce the harmonic voltage throughout the network as beneficial, while Toll Road pricing see all harmonic injections into the network as a burden.

6.6 CONCLUSION

The inclusion of a passive filter in the network was shown to change the marginal value of all injected harmonic currents. This value of injected currents is summarised in the Lagrange multiplier $\widetilde{\mu}_h$. For marginal harmonic prices to be efficient, they must reflect this change in value. Efficiency with respect to passive marginal filters is characterised by the marginal value to the network of the filter capacity being equal to the marginal operating and capital cost of the filter. This condition is identical to that of active filters and results in the harmonic voltage throughout the network being minimised. That is the optimal phase angle for the included filter is that which minimises the harmonic voltage, given the filter admittance magnitude. In cases where the phase angle of the filter admittance is constrained, this must lead to a reduction in system welfare. But if the installed filter has a large admittance, the loss in welfare associated with this constraint is small, because the harmonic voltages throughout the network fall to values close to zero, independent of the filter's phase.

While it is easy to characterise the ideal passive filter for the system, bringing about this optimal result through a market mechanism can be difficult. The previous technique of using $\widetilde{\mu}_h$ as the marginal prices was shown to have some deficiencies. Specifically these prices fail to collect the correct amount of revenue. Another form of marginal prices was developed ($\widetilde{\mu}_h^s$), that do collect the correct amount of revenue. But $\widetilde{\mu}_h^s$ ignores the presence of the passive filter and hence conveys inefficient signals to the market as to the value of injected harmonics.

Where the filter owner is given a large amount of market power (the ability to control the price), neither set of prices will encourage optimal behaviour with respect to the inclusion of passive filter capacity. Paradoxically under such conditions, the fact that the prices $\widetilde{\mu}_h^s$ are less responsive to the included filter capacity mean these prices will encourage more efficient behaviour, than the prices $\widetilde{\mu}_h$. But if the filter owners have little influence over the harmonic price (competitive harmonic market), the prices that accurately reflect the true value of harmonic injections ($\widetilde{\mu}_h$) will encourage the most efficient behaviour. The advantage of marginal pricing is that, in theory, by making the value of individual actions transparent, an efficient allocation of resources can be achieved. With respect to harmonic pricing, one of the goals of the pricing system must be to achieve an effiecient allocation of filter resources. It has been demonstrated unless the market is structured to restrict the abuse of market power, marginal pricing has diminished value. However the information which falls out of the marginal prices is "true", independent of if a market per se exists or not. As such the use of the marginal prices will be of great value to any authority that wishes to mitigate some harmonic problem via the use of a passive filter. The marginal prices will indicate what type of filter is optimal given the preferences of the loads in the network.

Restricting filters from abusing market power should be possible by paying filter owners an amount based on the amount of installed filter capacity. It is possible to develop marginal filter capacity prices, which have the same efficient properties as the marginal harmonic current prices. These prices for filter capacity are essentially a modified version of prices for current injections.

It was shown that $\widetilde{\mu}_h$ and hence the value to the network of injected harmonics is heavily dependent on the magnitude of any included filter capacity. As the size of any included filter increases, the value of injected harmonics rapidly decreases towards zero. On the other hand the prices $\widetilde{\mu}_h^s$, were shown to be largely independent of the magnitude of any installed filter. The consequence being $\widetilde{\mu}_h^s$, will encourage uneconomic filter capacity to be installed, which ultimately results in a reduction of aggregate network utility. Network utility will be maximised where the harmonic market is competitive and the prices $\widetilde{\mu}_h$ are used.

Chapter 7

DYNAMIC NONLINEAR DEVICES

7.1 INTRODUCTION

To this point nonlinear loads have been modelled as static harmonic current sources (Figure 7.1).

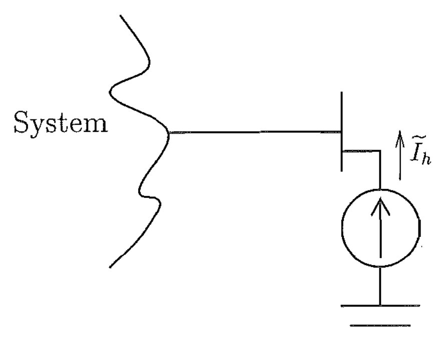


Figure 7.1 Static nonlinear load model

This model fails to accurately represent the true behaviour of most nonlinear loads. In particular it fails to recognise that the injected current from the nonlinear load will be a function of the voltage waveform at the busbar. If harmonic prices are to reflect the true marginal value of injected harmonic current into the network, they must account for the effect each injection has on all the other injections in the network.

One common problem arising from the standard based control of harmonic distortion is how to deal with non-characteristic harmonics produced by loads due to a distorted commutating voltage. In situations where, if given a pure fundamental commutating voltage, the nonlinear load meets the standards, but the voltage distortion results in injections that fail to comply with the standard, who should bear the costs of making the nonlinear load compliant can be a contentious issue. It is possible to incorporate rules into the harmonic standards to deal with all such possible situations, but these rules are likely to be rather arbitrary, and shaped by superior lobbying and influence. Such rules cannot produce an efficient outcome as an efficient outcome will require a different response for each situation.

This chapter looks at marginal pricing where the nonlinear loads are modelled as voltage dependent current sources or as Norton equivalents. An example of marginal pricing where the nonlinear loads are modelled as dynamic sources is included. Also active and passive filters are included in this new pricing environment.

7.2 ADJUSTED MARGINAL PRICES

7.2.1 Nonlinear Loads: Voltage Dependent Current Sources

One alternative for modelling the nonlinear loads is to use a voltage dependent current source, where the harmonic injections are given by:

$$\text{Harmonic current injections into the network by load } i = \widetilde{I}_{hi} = f(\widetilde{V}_{hi}) \tag{7.1}$$

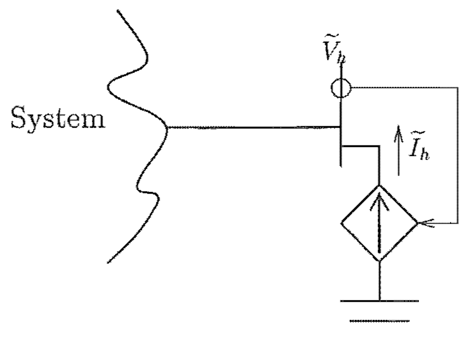


Figure 7.2 Voltage dependent current source model of nonlinear loads

With such a model the harmonic current injected into the network will be dependent on both the phase angle and magnitude of the harmonic voltage seen at the local busbar.

$$I_{hi} = f_{mag}(V_{hi}, \theta_i) \tag{7.2}$$
$$\alpha_i = f_{ang}(V_{hi}, \theta_i) \tag{7.3}$$

In modelling such devices tensors must be used. There are three possible candidates [Smith1996] [Daza1988].

$$\text{Rectangular} \quad \begin{bmatrix} I_{hr} \\ I_{hj} \end{bmatrix} = \begin{bmatrix} f_r\,(V_{hr}, V_{hj}) \\ f_j\,(V_{hr}, V_{hj}) \end{bmatrix}$$
$$\text{Conjugate} \quad \begin{bmatrix} \widetilde{I}_h \\ \widetilde{I}_h^* \end{bmatrix} = \begin{bmatrix} f_1\left(\widetilde{V}_h, \widetilde{V}_h^*\right) \\ f_2\left(\widetilde{V}_h, \widetilde{V}_h^*\right) \end{bmatrix}$$
$$\text{Polar} \quad \begin{bmatrix} I_h \\ \alpha \end{bmatrix} = \begin{bmatrix} f_I\,(V_h, \theta) \\ f_\alpha\,(V_h, \theta) \end{bmatrix}$$

Modelling the nonlinear devices in the polar domain is attractive, as the magnitude of harmonic distortion is the variable of interest. Network circuit analysis in the polar domain ceases to be linear, ruling it out as an alternative. As such working in the conjugate harmonic space is the preffered option as it allows differentiation by the voltage magnitude, and circuit analysis is linear.

Working in this domain the network nodal equation can be rewritten as:

$$\begin{bmatrix} \widetilde{\mathbf{I}}_\mathbf{h} \\ \widetilde{\mathbf{I}}_\mathbf{h}^* \end{bmatrix} = \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_\mathbf{h} \\ \widetilde{\mathbf{V}}_\mathbf{h}^* \end{bmatrix} \tag{7.4}$$

Where Y_h = The network admittance matrix, which previously was specified as $[Y_h]$

Using the network nodal equation as the sole constraint (at this point ignore the ability of loads to reduce their injections, and the possible inclusion of filters), the utility maximisation problem for the aggregate network can be solved using the following Lagrangian:

$$\mathcal{L} = \begin{bmatrix} U(\mathbf{V_h}) \\ U^*(\mathbf{V_h}) \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} \widetilde{\mathbf{I}}_\mathbf{h}(\widetilde{\mathbf{V}}_\mathbf{h}, \widetilde{\mathbf{V}}_\mathbf{h}^*) \\ \widetilde{\mathbf{I}}_\mathbf{h}^*(\widetilde{\mathbf{V}}_\mathbf{h}, \widetilde{\mathbf{V}}_\mathbf{h}^*) \end{bmatrix} - \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_\mathbf{h} \\ \widetilde{\mathbf{V}}_\mathbf{h}^* \end{bmatrix} \right) \tag{7.5}$$

The first order condition associated with this problem is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} &= \begin{bmatrix} \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \\ \frac{\partial U^*(\mathbf{V_h})}{\partial \mathbf{V_h}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} + \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \\ \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}^*}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} + \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}^*}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \end{bmatrix} - \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} \\ \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \\ \frac{\partial U^*(\mathbf{V_h})}{\partial \mathbf{V_h}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} - Y_h & \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} \\ \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}^*}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} & \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}^*}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} - Y_h^* \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} \\ \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \end{bmatrix} \right) = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{7.6}$$

The term $\widetilde{\mu}_N$ can be solved for using the first equation in matrix Equation 7.6.

$$\begin{aligned} \widetilde{\mu}_N &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \left(\begin{bmatrix} Y_h - \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} & -\frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} \\ \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \end{bmatrix} \right)^{-1} \\ &= \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \left(\begin{bmatrix} Y_h - \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}} & -\frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*} \end{bmatrix} \begin{bmatrix} [e^{j\theta}] \\ [e^{-j\theta}] \end{bmatrix} \right)^{-1} \end{aligned} \tag{7.7}$$

Given that $\widetilde{\mathbf{I}}_\mathbf{h}$ is a function of the voltage, it is not all together clear what the Lagrange multiplier $\widetilde{\mu}_N$ represents. As such it is of value to look at the consequences of modelling the nonlinear load as a Norton equivalent.

7.2.2 Nonlinear Loads: Norton Equivalent

For small perturbations around the operating point (at the margin), the Norton equivalent, and voltage dependent source models look identical [Bathurst1999]. The Norton representation of the nonlinear load is shown in Figure 7.3.

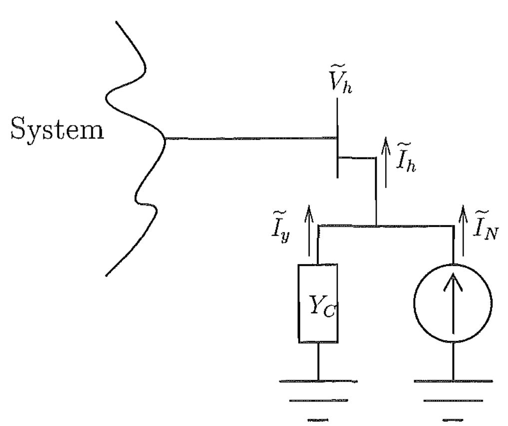


Figure 7.3 Norton equivalent model of nonlinear loads

To come up with the terms of Y_C, the device is linearised around its operating point.

$$\begin{bmatrix} \Delta\widetilde{\mathbf{I}}_{\mathbf{h}} \\ \Delta\widetilde{\mathbf{I}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix} \begin{bmatrix} \Delta\widetilde{\mathbf{V}}_{\mathbf{h}} \\ \Delta\widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} \tag{7.8}$$

$$\text{Where} \quad Y_1 = \frac{\partial\widetilde{\mathbf{I}}_{\mathbf{h}}}{\partial\widetilde{\mathbf{V}}_{\mathbf{h}}} \tag{7.9}$$

$$Y_2 = \frac{\partial\widetilde{\mathbf{I}}_{\mathbf{h}}}{\partial\widetilde{\mathbf{V}}^*_{\mathbf{h}}} \tag{7.10}$$

Therefore for the purposes of the Norton equivalent:

$$\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{y}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} \tag{7.11}$$

The Norton injection $\widetilde{\mathbf{I}}_{\mathbf{N}}$, is then the residual given by:

$$\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{h}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{y}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} \tag{7.12}$$

When using the Norton equivalents, the nodal equation takes the form:

$$\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{y}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{y}} \end{bmatrix} - \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} Y_1 - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{7.13}$$

Given equation 7.13, the constrained utility maximisation problem can be solved using the following Lagrangian.

$$\mathcal{L} = \begin{bmatrix} U(\mathbf{V}_{\mathbf{h}}) \\ U^*(\mathbf{V}_{\mathbf{h}}) \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} Y_1 - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} \right) \tag{7.14}$$

The associated first order condition is:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{V}_{\mathbf{h}}} = \begin{bmatrix} \frac{\partial U(\mathbf{V}_{\mathbf{h}})}{\partial\mathbf{V}_{\mathbf{h}}} \\ \frac{\partial U^*(\mathbf{V}_{\mathbf{h}})}{\partial\mathbf{V}_{\mathbf{h}}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} Y_1 - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_h^* \end{bmatrix} \begin{bmatrix} \frac{\partial\widetilde{\mathbf{V}}_{\mathbf{h}}}{\partial\mathbf{V}_{\mathbf{h}}} \\ \frac{\partial\widetilde{\mathbf{V}}^*_{\mathbf{h}}}{\partial\mathbf{V}_{\mathbf{h}}} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.15}$$

$$\Rightarrow \widetilde{\mu}_N = \frac{\partial U(\mathbf{V}_{\mathbf{h}})}{\partial\mathbf{V}_{\mathbf{h}}} \left(\begin{bmatrix} Y_h - Y_1 & -Y_2 \end{bmatrix} \begin{bmatrix} [e^{j\theta}] \\ [e^{-j\theta}] \end{bmatrix} \right)^{-1} \tag{7.16}$$

This is exactly the same formulation for $\widetilde{\mu}_N$ as Equation 7.7. Working within the Norton framework, $\widetilde{\mu}_N$ is easily seen to represent the marginal value to the system of Norton injections into the network, as:

$$\widetilde{\mu}_N = \frac{\partial U}{\partial\widetilde{\mathbf{I}}_{\mathbf{N}}} \tag{7.17}$$

7.2.3 Norton Injection Prices to Harmonic Injection Prices

By modelling the nonlinear loads as either voltage dependent current sources or Norton equivalents, it is possible to develop the marginal price for Norton injections into the network. But the Norton injections are a mythical mathematical construct, charging loads on the basis of an imaginary quantity is not ideal. It would be preferable to use marginal prices for the actual harmonic injections ($\widetilde{\mathbf{I}}_\mathrm{h}$), which reflect the value of the actual harmonic injections into the network. It is possible to convert the prices $\widetilde{\mu}_N$, into such a value.

Looking at Figure 7.3, and ignoring the fact the variables must be described via tensors.

$$\begin{aligned}
\widetilde{\mathbf{I}}_\mathrm{y} &= [Y_C]\widetilde{\mathbf{V}}_\mathrm{h} && (7.18)\\
\text{Where}\quad \widetilde{\mathbf{V}}_\mathrm{h} &= [Y_T]^{-1}\widetilde{\mathbf{I}}_\mathrm{N} && (7.19)\\
[Y_T] &= [Y_h] - [Y_C] && (7.20)\\
\therefore \widetilde{\mathbf{I}}_\mathrm{y} &= [Y_C][Y_T]^{-1}\widetilde{\mathbf{I}}_\mathrm{N} && (7.21)\\
\text{So that}\quad \widetilde{\mathbf{I}}_\mathrm{h} &= [Y_C][Y_T]^{-1}\widetilde{\mathbf{I}}_\mathrm{N} + \widetilde{\mathbf{I}}_\mathrm{N} \\
&= \left([Y_C][Y_T]^{-1} + [I]\right)\widetilde{\mathbf{I}}_\mathrm{N} && (7.22)
\end{aligned}$$

Looking at the marginal value of the Norton injections.

$$\begin{aligned}
\frac{\partial U}{\partial \widetilde{\mathbf{I}}_\mathrm{N}} &= \frac{\partial U}{\partial \widetilde{\mathbf{I}}_\mathrm{h}}\frac{\partial \widetilde{\mathbf{I}}_\mathrm{h}}{\partial \widetilde{\mathbf{I}}_\mathrm{N}} \\
&= \frac{\partial U}{\partial \widetilde{\mathbf{I}}_\mathrm{h}}\left([Y_C][Y_T]^{-1} + [I]\right) \\
\Rightarrow \frac{\partial U}{\partial \widetilde{\mathbf{I}}_\mathrm{h}} &= \frac{\partial U}{\partial \widetilde{\mathbf{I}}_\mathrm{N}}\left([Y_C][Y_T]^{-1} + [I]\right)^{-1} \\
\widetilde{\mu}_T &= \widetilde{\mu}_N\left([Y_C][Y_T]^{-1} + [I]\right)^{-1}
\end{aligned} \qquad (7.23)$$

The above result can be extended to the tensor environment in which the nonlinear device must be described. Using the following definitions:

$$\widetilde{\mathbf{II}}_\mathrm{h} = \begin{bmatrix} \widetilde{\mathbf{I}}_\mathrm{h} \\ \widetilde{\mathbf{I}}^*_\mathrm{h} \end{bmatrix} \qquad (7.24)$$

$$\widetilde{\mathbf{II}}_\mathrm{N} = \begin{bmatrix} \widetilde{\mathbf{I}}_\mathrm{N} \\ \widetilde{\mathbf{I}}^*_\mathrm{N} \end{bmatrix} \qquad (7.25)$$

Equation 7.23, can be extended to a tensor equivalent.

$$\frac{\partial U}{\partial \widetilde{\mathbf{II}}_\mathrm{h}} = \frac{\partial U}{\partial \widetilde{\mathbf{II}}_\mathrm{N}}\left([I] + \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix}\begin{bmatrix} Y_h - Y_1 & -Y_2 \\ -Y_2^* & Y_h^* - Y_1^* \end{bmatrix}^{-1}\right)^{-1} \qquad (7.26)$$

From the constraint shown in Equation 7.13, it can be seen:

$$\frac{\partial U}{\partial \widetilde{\mathbf{II}}_\mathrm{N}} = \begin{bmatrix} \widetilde{\mu}_N & \widetilde{\mu}^*_N \end{bmatrix} \qquad (7.27)$$

As such working within the tensor framework, the marginal prices for harmonic injections can be derived from the marginal prices for Norton injections using:

$$\left[\begin{array}{cc} \widetilde{\mu}_T & \widetilde{\mu}_T^* \end{array}\right] = \left[\begin{array}{cc} \widetilde{\mu}_N & \widetilde{\mu}_N^* \end{array}\right] \left([I] + \left[\begin{array}{cc} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{array}\right] \left[\begin{array}{cc} Y_h - Y_1 & -Y_2 \\ -Y_2^* & Y_h^* - Y_1^* \end{array}\right]^{-1} \right)^{-1} \tag{7.28}$$

From Equation 7.28 the desired marginal prices for harmonic injections, $\widetilde{\mu}_T$, can be extracted.

7.2.4 Revenue Reconciliation

In the case where each loads' utility function is a linear function of the harmonic voltage magnitude ($u_i(V_{hi}) = k_i V_{hi}\ \forall i$), the total amount that needs to be collected from the injectors of harmonic currents is:

$$\text{Total harmonic compensation payments} = \mathbf{K}\mathbf{V_h} \tag{7.29}$$

The question whether if the harmonic prices $\widetilde{\mu}_T$, calculated above in Equation 7.28, collect the correct amount from the nonlinear loads? Given the complexity of $\widetilde{\mu}_T$, direct calculation of $\widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{h}$, is difficult. Instead consider the amount collected from Norton injections into the network, if harmonic charges were based on such injections.

$$\begin{aligned} \widetilde{\mu}_N &= \mathbf{K}\left(\left[\begin{array}{cc} Y_h - Y_1 & -Y_2 \end{array}\right] \left[\begin{array}{c} [e^{j\theta}] \\ [e^{-j\theta}] \end{array}\right] \right)^{-1} \\ &= \mathbf{K}\left([Y_h - Y_1][e^{j\theta}] - [Y_2][e^{-j\theta}] \right)^{-1} \end{aligned} \tag{7.30}$$

$$\begin{aligned} \widetilde{\mathbf{I}}_\mathbf{N} &= \left[\begin{array}{cc} Y_h - Y_1 & -Y_2 \end{array}\right] \left[\begin{array}{c} \widetilde{\mathbf{V}}_\mathbf{h} \\ \widetilde{\mathbf{V}}_\mathbf{h}^* \end{array}\right] \\ &= \left([Y_h - Y_1][e^{j\theta}] - [Y_2][e^{-j\theta}] \right) \mathbf{V_h} \end{aligned} \tag{7.31}$$

$$\Rightarrow \widetilde{\mu}_N \widetilde{\mathbf{I}}_\mathbf{N} = \mathbf{K}\mathbf{V_h} \tag{7.32}$$

Equations 7.30, 7.31 and 7.32, show that if loads were charged $\widetilde{\mu}_N$ for their Norton injections, the required amount would be collected.

$$\text{Given that} \quad \widetilde{\mu}_N = \widetilde{\mu}_T \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{I}}_\mathbf{N}} \tag{7.33}$$

$$\Rightarrow \widetilde{\mu}_T \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{I}}_\mathbf{N}} \widetilde{\mathbf{I}}_\mathbf{N} = \mathbf{K}\mathbf{V_h} \tag{7.34}$$

As $\widetilde{\mathbf{I}}_\mathbf{h}$ is linear in $\widetilde{\mathbf{I}}_\mathbf{N}$ (equation 7.22)

$$\frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \widetilde{\mathbf{I}}_\mathbf{N}} \widetilde{\mathbf{I}}_\mathbf{N} = \widetilde{\mathbf{I}}_\mathbf{h} \tag{7.35}$$

Combining the above two equations produces the required result:

$$\widetilde{\mu}_T \widetilde{\mathbf{I}}_\mathbf{h} = \mathbf{K}\mathbf{V}_\mathbf{h} \tag{7.36}$$

Therefore, the adjusted prices for total harmonic injections will collect the required amount from the polluting loads.

7.3 EXAMPLE

In this example the test system of Appendix A is used. The behaviour of the prices for Norton ($\widetilde{\mu}_N$), and harmonic injections ($\widetilde{\mu}_T$) are investigated, as the parameters of the impedance tensor for the load at busbar four are varied. In this example one of the nonlinear load's polar parameters is varied. The sensitivity of the harmonic injections from the nonlinear load at busbar four to the harmonic voltage seen at that busbar can be described via:

$$\begin{bmatrix} \Delta I_{h4} \\ \Delta \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial I_{h4}}{\partial V_{h4}} & \frac{\partial I_{h4}}{\partial \theta_4} \\ \frac{\partial \alpha_4}{\partial V_{h4}} & \frac{\partial \alpha_4}{\partial \theta_4} \end{bmatrix} \begin{bmatrix} \Delta V_{h4} \\ \Delta \theta_4 \end{bmatrix} \tag{7.37}$$

In this example the sensitivity of injected current magnitude, to changes in harmonic voltage magnitude ($\frac{\partial I_{h4}}{\partial V_{h4}}$) is varied. All the other parameters are set equal to zero.

$$\begin{bmatrix} \Delta I_{h4} \\ \Delta \alpha_4 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{h4} \\ \Delta \theta_4 \end{bmatrix} \quad \text{Where } k = 0 \to 20 \tag{7.38}$$

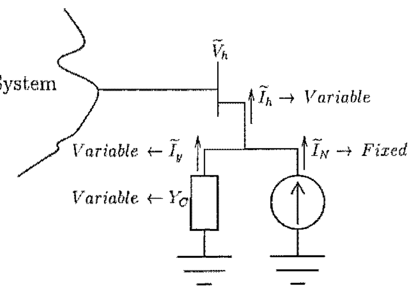
This variation in the load's polar parameters, is easily transformed to variation in its complex conjugate tensor parameters, using the transform detailed in Appendix E. This transform is required to keep the network nodal equation linear. All the other loads are modelled as static harmonic current sources, as before.

7.3.1 Constant Norton Injection

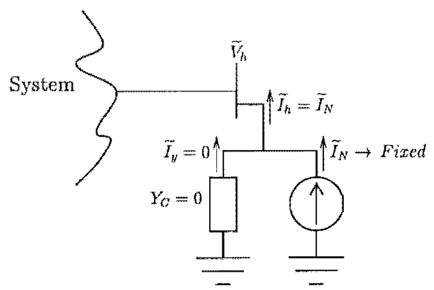
The injected harmonic currents detailed in Appendix A are specified as the Norton injections for each load. Except for the nonlinear load at busbar 4, these Norton injections are their total injections for each load. The nonlinear load at busbar four is modelled as a Norton equivalent, where the impedance parameters are varied (as described above). The result being the Norton injection for each the loads is constant, but the total harmonic injection from the load at busbar four will vary with the change in that load's tensor impedance (Figure 7.4).

As all the nonlinear loads apart from that at busbar four are modelled as static, their harmonic injections are constant as shown Figure 7.5. It can be seen that the harmonic injections for the load at busbar four vary considerably as the parameters of it's Norton impedance are varied. Specifically, there is a resonance at the point where $\frac{\partial I_{h4}}{\partial V_{h4}}$ approximately equals 10). This resonance in the injected harmonic current from the load at busbar four, is accompanied by a similar resonance in the harmonic voltage at each busbar as shown in Figure 7.6. As the network is strong, the voltage at each busbar is very similar, this was also seen in previous examples.

Figure 7.7 shows the magnitude and angle of the prices for Norton injections as the parameters of the nonlinear load at busbar four are varied. The resonance in the harmonic current and voltage, is associated with a resonance in the marginal price for Norton injections. This is expected as at this resonance point, the Norton injections from all the loads are contributing to a distortion voltage at busbar four, which in turn is producing a very large (in phase) harmonic injection from the local nonlinear load.

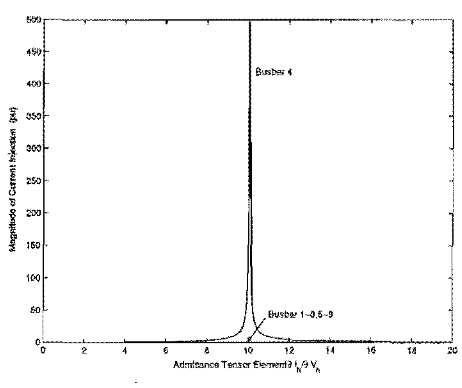


(a) Busbar four

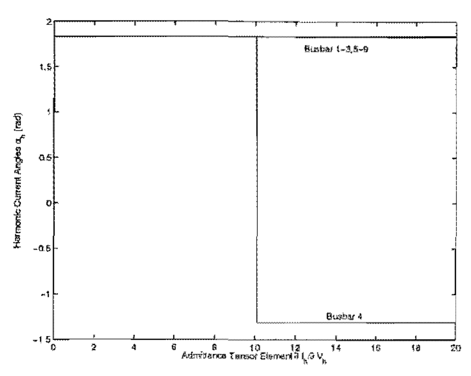


(b) All other busbars

Figure 7.4 Norton equivalent models of the nonlinear loads used in the example

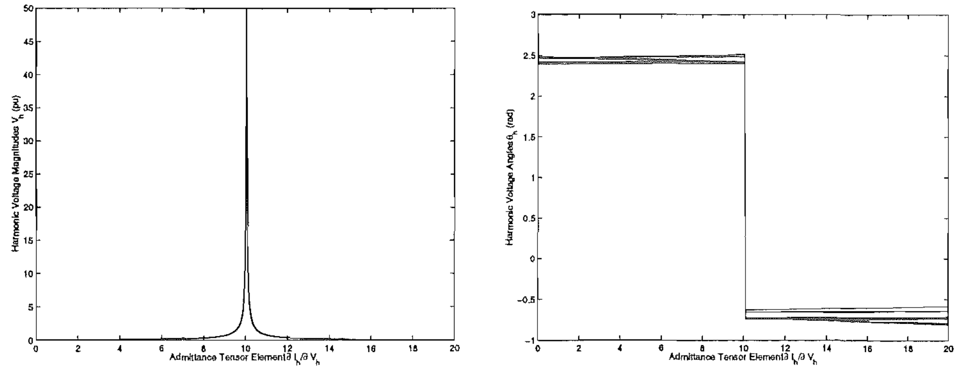


(a) Current Magnitude



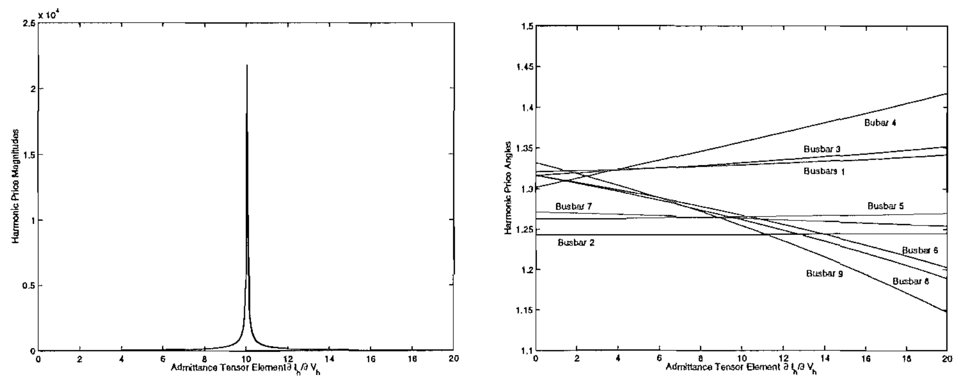
(b) Current Angle

Figure 7.5 Magnitude and angle of harmonic injections from the loads as parameters of busbar four are varied



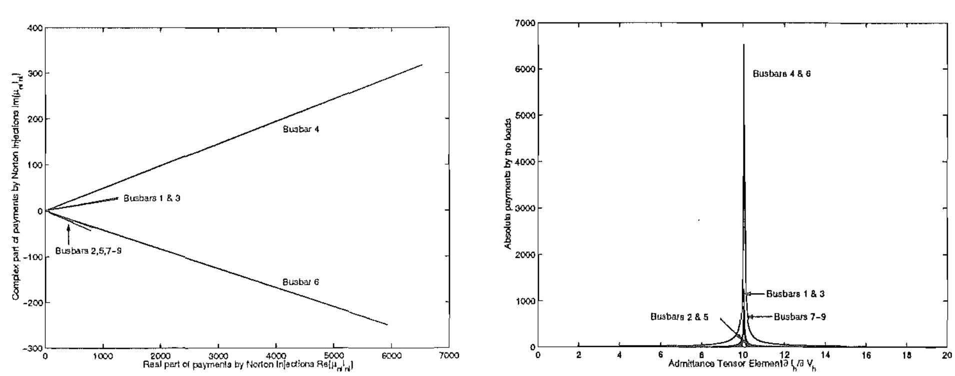
(a) Voltage Magnitude (b) Voltage Angle

Figure 7.6 Magnitude and angle of harmonic voltage at each busbar as parameters of busbar four are varied



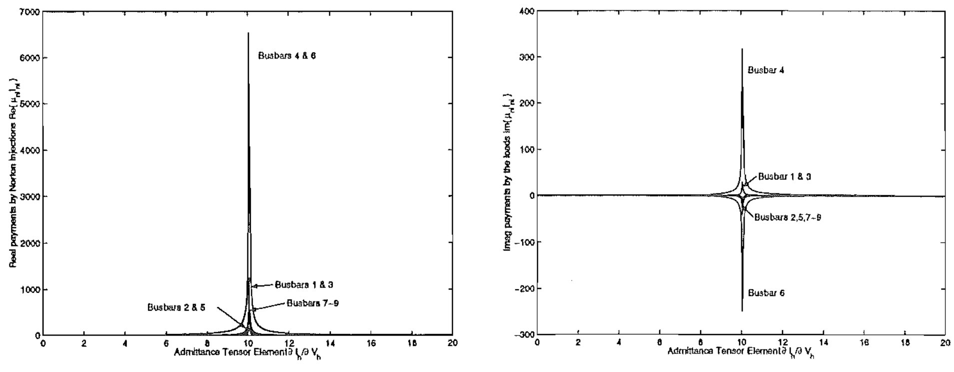
(a) Price Magnitude μ_N (b) Price Angle $\angle\widetilde{\mu}_N$

Figure 7.7 Magnitude and angle of the prices for Norton injections as parameters of busbar four are varied



(a) $\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\ \forall i$ (b) $||\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}||\ \forall i$

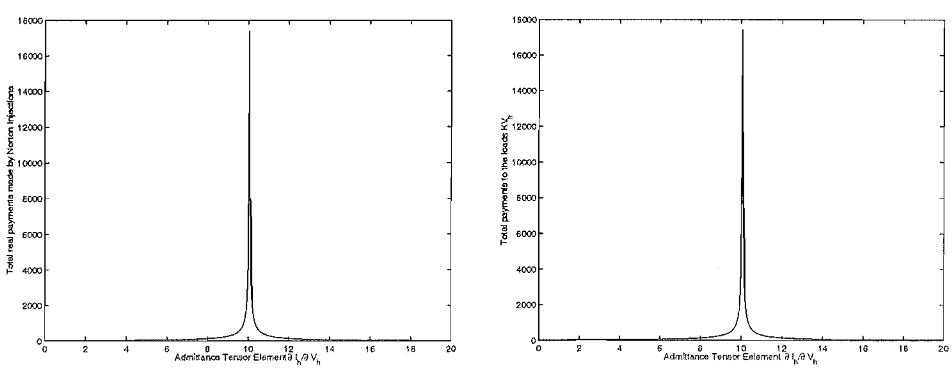
Figure 7.8 Complex payments due from nonlinear loads on the basis of Norton injections, and the magnitude of those payments



(a) $Re\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}\ \forall i$ (b) $Im\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}\ \forall i$

Figure 7.9 Real and imaginary components of payments due from nonlinear loads on the basis of Norton injections and prices

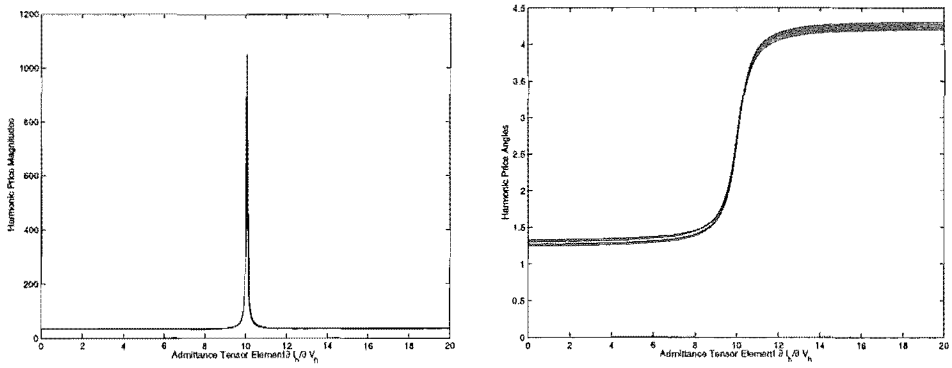
Figures 7.8 and 7.9 show the Norton payments due from each load. That is, they show the payments each load would face if charged on the basis of Norton injections, using the calculated price for such injections ($\widetilde{\mu}_N$). In general these payments are complex and, as before, it is only the real part of these payments that actually represent the cost associated with each load's injections, and hence it is only the real part of these charges each load needs to pay to achieve optimality. Because the Norton injections from all the loads have the same phase, and hence affect the distortion voltage at busbar four in a similar manner, each load faces a similar charge for their Norton injections. The relative real payments made by each load are therefore just a reflection of the magnitude of each loads Norton injection as detailed in the appendix. One can see that the payments due from each load while complex are mainly real, this is a consequence of the fact that all the Norton injections are at a common angle, and the network is strong. Naturally, at the voltage resonance point, there is also a resonance in the value of the Norton injections.



(a) $Re\{\widetilde{\mu}_N\widetilde{I}_N\}$ (b) $\mathbf{KV_h}$

Figure 7.10 Total real payments due for Norton injections, and the total harmonic compensation payments due to loads

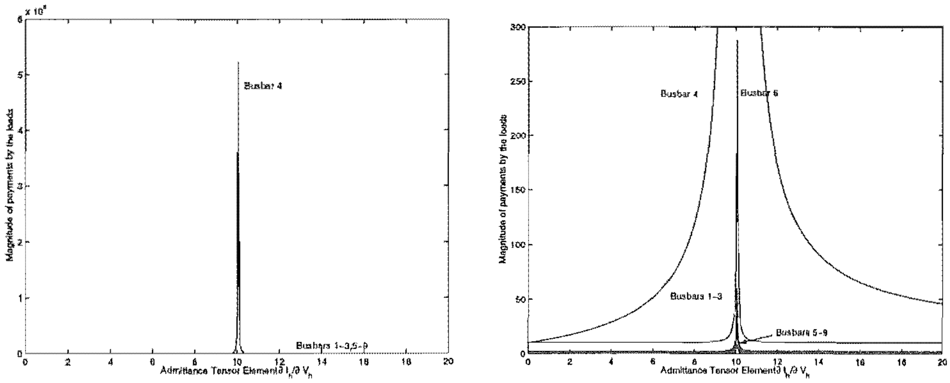
It was shown that if loads were to be charged based on their Norton injections, the amount collected for these injections ($Re\{\widetilde{\mu}_N\widetilde{I}_N\}$) would equal the compensation payments due for distortion seen at each busbar ($\mathbf{KV_h}$) (assuming loads have a right to a clean voltage supply). This holds true in this example as shown in Figure 7.10.



(a) Price Magnitude μ_T (b) Price Angle $\angle\widetilde{\mu}_T$

Figure 7.11 Magnitude and angle of the prices for harmonic injections as parameters of busbar four are varied

Figure 7.11, shows the variation in the price for harmonic injections as the parameters at busbar four are varied. The magnitude of these prices, not surprisingly behave in a similar manner to the prices for Norton injections. At the resonance point, the phase angle of the harmonic price at each busbar shifts.

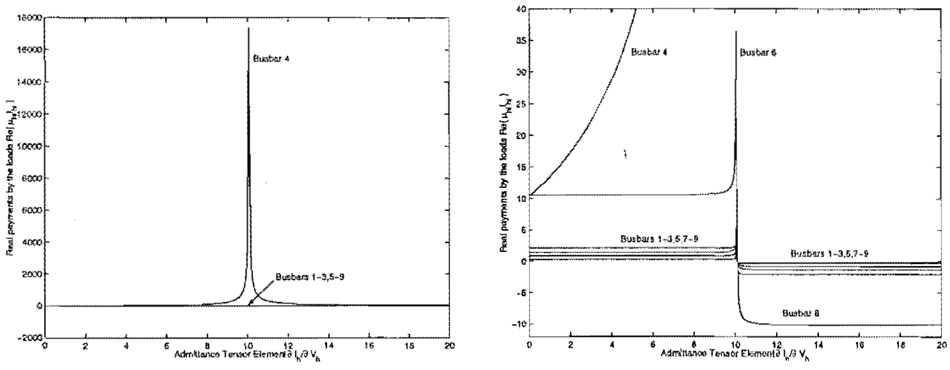


(a) $\|\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\|\ \forall i$ (b) $\|\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\|\ \forall i$

Figure 7.12 Absolute value of the complex payments due for the harmonic injections

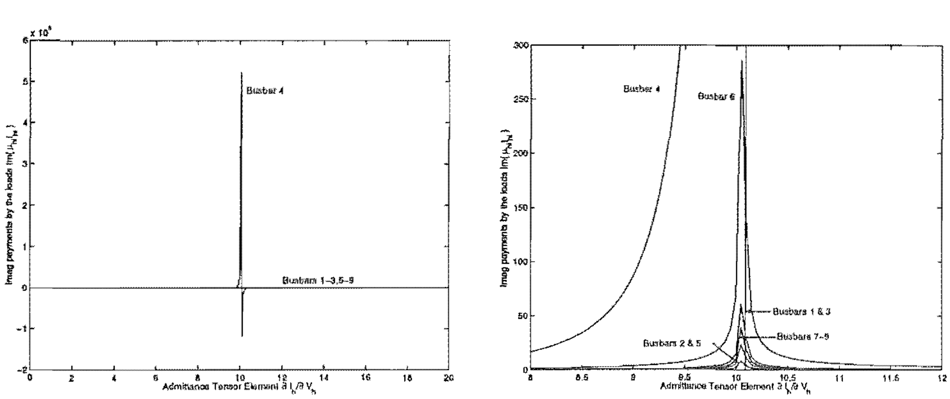
Figures 7.12 through 7.14, show the variation in the harmonic payments due from each load as the parameters of the nonlinear load at busbar four are varied, when charging loads on the basis of their harmonic injections ($\widetilde{\mu}_T$ used). Of particular interest is the real component of the complex payments, as they represent the true value/cost of the harmonic injections. The first point to note is that the vast majority of the harmonic payments are due from the load at busbar four. This is expected given the majority of the injections into the network are from this load. The shift in the phase of the harmonic voltage (Figure 7.6) suggests the non-Norton current injected into the network is out of phase with the Norton injections of each load. This shift in the voltage phase angle is therefore associated with the Norton injections of the other loads changing from being being the dominant cause of voltage distortion, to less influential than $\widetilde{I}_{Y4}$. Figure 7.13 shows that at the resonance point, the loads apart from that at busbar four go from being charged for their injections, to being rewarded as a result of this shift in voltage angle.

Figure 7.15, shows the total real amount charged to the injectors of harmonic current and



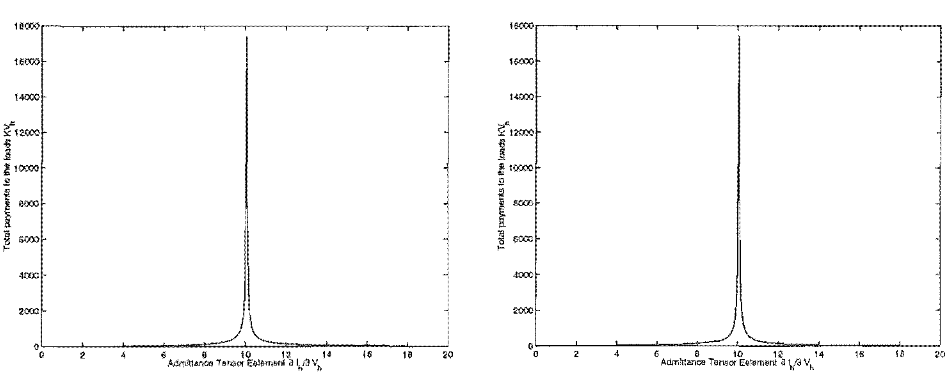
(a) $Re\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}\ \forall i$ (b) $Re\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}\ \forall i$

Figure 7.13 Real component of the complex payments due for the harmonic injections



(a) $Im\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}\ \forall i$ (b) $Im\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}\ \forall i$

Figure 7.14 Imaginary component of the complex payments due for the harmonic injections



(a) $Re\{\widetilde{\mu}_{T}\widetilde{I}_{h}\}$ (b) $\mathbf{KV_h}$

Figure 7.15 Total real payments due for harmonic injections, and the total harmonic compensation payments due to loads the total amount due to loads as compensation for the distortion seen at the local busbar. As suggested the adjusted prices for harmonic injections $\widetilde{\mu}_T$, will collect the correct amount.

The defining feature in all the previous figures is the resonance at $\frac{\partial I_{h4}}{\partial V_{h4}} \approx 10$. The resonance occurs as at the point where the total system ceases to be passive (Appendix F). Figure 7.16 shows the admittance loci looking into busbar 4, for different values of $\frac{\partial I_{h4}}{\partial V_{h4}}$. These loci form a series of concentric circles. As $\frac{\partial I_{h4}}{\partial V_{h4}}$, increases the radius of the admittance loci increases, to the point where for some voltages the real part of the admittance looking into the system is negative. Such a situation is artifical, as power electronic loads are passive.

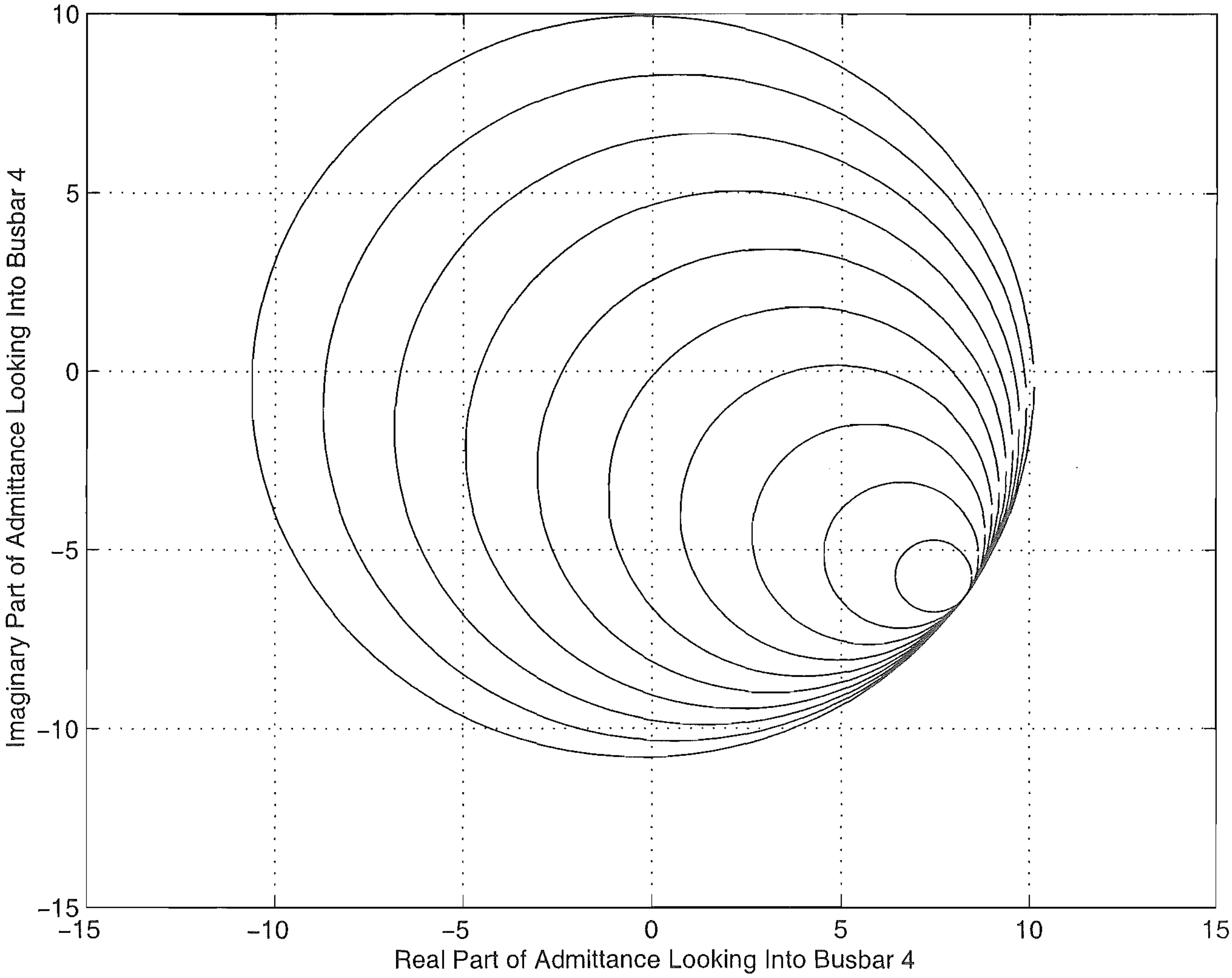
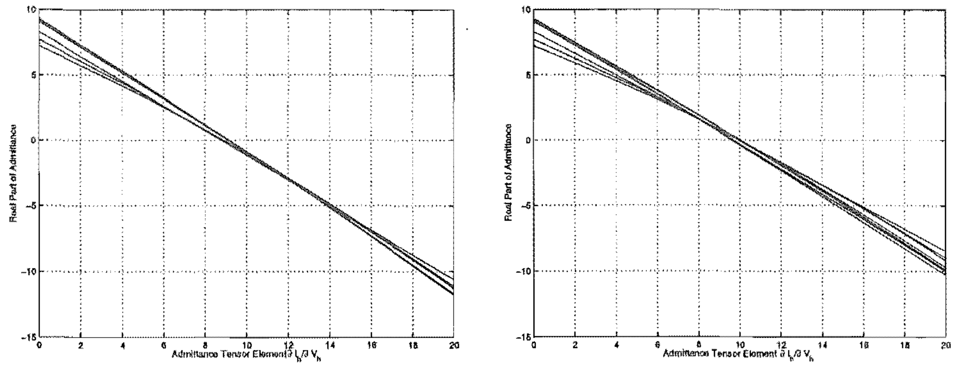


Figure 7.16 Admittance loci looking into busbar four as the parameters of the local nonlinear load are varied. Specifically as $\frac{\partial I_{h4}}{\partial V_{h4}} = 2 \rightarrow 20$ in steps of two

Figure 7.17, shows the minimum possible real component of the admittance looking into each busbar, along with the actual real component of the admittance looking into each busbar. That is, the two diagrams show the earliest possible point at which the system could cease to be passive, and the actual point at which it ceases to be passive. As the network is strong, the impedance looking in at each busbar is approximately equal. It can be seen that at the resonance point ($\frac{\partial I_{h4}}{\partial V_{h4}} \approx 10$) the system switches to from being active to passive.

Another example is detailed in Appendix G. The example is identical to the one above, except the total harmonic injection of each load is fixed, with the Norton impedance and current at busbar four both variable.



(a) Minimum Possible Real Component

(b) Actual Real Component

Figure 7.17 The minimum possible, and actual real components of the admittance looking into busbar four

7.4 ACTIVE FILTER IN A TENSOR ENVIRONMENT

Previously in Chapter 5, active filters where modelled as a controllable current source. This is an adequate representation that will now be used within the tensor environment.

To incorporate the presence of active filters into the utility optimisation problem within this tensor environment one needs to specify some quantities:

$$\text{Active filter injections into the network} = \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{f}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{f}} \end{bmatrix} \tag{7.39}$$

$$\text{Operating cost of active filter injections} = \begin{bmatrix} FV(\mathbf{I_f}) \\ FV^*(\mathbf{I_f}) \end{bmatrix} \tag{7.40}$$

$$\text{Capital cost of active filter} = \begin{bmatrix} FF(\mathbf{I_{fmax}}) \\ FF^*(\mathbf{I_{fmax}}) \end{bmatrix} \tag{7.41}$$

The utility maximisation where active filters are included can be described by:

$$\text{Maximise} \quad \begin{bmatrix} U(\mathbf{V_h}) \\ U^*(\mathbf{V_h}) \end{bmatrix} - \begin{bmatrix} FV(\mathbf{I_f}) \\ FV^*(\mathbf{I_f}) \end{bmatrix} - \begin{bmatrix} FF(\mathbf{I_{fmax}}) \\ FF^*(\mathbf{I_{fmax}}) \end{bmatrix} \tag{7.42}$$

$$\text{Subject to} \quad \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{h}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{h}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{f}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{f}} \end{bmatrix} - \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} \mathbf{I_f} \\ \mathbf{I_f} \end{bmatrix} \leq \begin{bmatrix} \mathbf{I_{fmax}} \\ \mathbf{I_{fmax}} \end{bmatrix} \tag{7.44}$$

The Lagrangian associated with this problem is:

$$\begin{aligned} \mathcal{L} = & \begin{bmatrix} U(\mathbf{V_h}) \\ U^*(\mathbf{V_h}) \end{bmatrix} - \begin{bmatrix} FV(\mathbf{I_f}) \\ FV^*(\mathbf{I_f}) \end{bmatrix} - \begin{bmatrix} FF(\mathbf{I_{fmax}}) \\ FF^*(\mathbf{I_{fmax}}) \end{bmatrix} \\ & + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}^*_N \end{bmatrix} \left(\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{f}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{f}} \end{bmatrix} + \begin{bmatrix} Y_1 - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} \right) \\ & + \begin{bmatrix} \lambda_f & 0 \\ 0 & \lambda_f^* \end{bmatrix} \left(\begin{bmatrix} \mathbf{I_{fmax}} \\ \mathbf{I_{fmax}} \end{bmatrix} - \begin{bmatrix} \mathbf{I_f} \\ \mathbf{I_f} \end{bmatrix} \right) \end{aligned} \tag{7.45}$$

The first order conditions associated with this Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \begin{bmatrix} \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \\ \frac{\partial U^*(\mathbf{V_h})}{\partial \mathbf{V_h}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \left(\begin{bmatrix} Y_1 - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_h^* \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}}{\partial \mathbf{V_h}} \\ \frac{\partial \widetilde{\mathbf{V}}_\mathbf{h}^*}{\partial \mathbf{V_h}} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.46}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_f}} = - \begin{bmatrix} \frac{\partial FV(\mathbf{I_f})}{\partial \mathbf{I_f}} \\ \frac{\partial FV^*(\mathbf{I_f})}{\partial \mathbf{I_f}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}_N^* \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}}{\partial \mathbf{I_h}} \\ \frac{\partial \widetilde{\mathbf{I}}_\mathbf{h}^*}{\partial \mathbf{I_h}} \end{bmatrix} - \begin{bmatrix} \lambda_f \\ \lambda_f^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.47}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_{fmax}}} = - \begin{bmatrix} \frac{\partial FF(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} \\ \frac{\partial FF^*(\mathbf{I_{fmax}})}{\partial \mathbf{I_{fmax}}} \end{bmatrix} + \begin{bmatrix} \lambda_f \\ \lambda_f^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.48}$$

Equation 7.46 is the exact same expression as that used to develop the prices for the Norton injections (equation 7.16). This indicates the inclusion of an active filter has no 'structural' affects on the value of Norton injections. Equations 7.47 and 7.48 can be combined to indicate the optimal level of active filter investment.

$$\widetilde{\mu}_N[e^{j\psi}] = \frac{\partial FV\left(\mathbf{I_f}\right)}{\partial \mathbf{I_f}} + \frac{\partial FF\left(\mathbf{I_{fmax}}\right)}{\partial \mathbf{I_{fmax}}} \tag{7.49}$$

Equation 7.49, specifies that the optimal level of active filter capacity results in the value of the injection (as specified by $\widetilde{\mu}_N$), being equal to the marginal costs of providing the injection. Note that as an active filter is modelled as a current source, the value of it's injections are those of a Norton injection.

While equations 7.46 to 7.48 specify the optimal allocation of active filter resources in this tensor environment, of interest is can this allocation be achieved using a market mechanism? In answering this consideration must be made of what set of prices are used. Or more specifically should loads be charged for their Norton injections or harmonic injections. Given that the Norton injections are an imaginary construct, it would seem natural to charge loads on the basis of their harmonic injections. Thus the problem faced by the potential filter owner can be described as:

$$\text{Maximise} \quad \widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{f} - FV\left(\mathbf{I_f}\right) - FF\left(\mathbf{I_{fmax}}\right) \tag{7.50}$$

$$\text{Subject to} \quad \mathbf{I_f} \leq \mathbf{I_{fmax}} \tag{7.51}$$

Where the price for harmonic injections ($\widetilde{\mu}_T$) is derived from $\widetilde{\mu}_N$, according to equation 7.28.

The associated Lagrangian and first order condition for this problem are:

$$\mathcal{L} = \widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{f} - FV\left(\mathbf{I_f}\right) - FF\left(\mathbf{I_{fmax}}\right) + \lambda_f\left(\mathbf{I_{fmax}} - \mathbf{I_f}\right) \tag{7.52}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_f}} = \widetilde{\mu}_T[e^{j\psi}] - \frac{\partial FV\left(\mathbf{I_f}\right)}{\partial \mathbf{I_f}} - \lambda_f = 0 \tag{7.53}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I_{fmax}}} = -\frac{\partial FF\left(\mathbf{I_{fmax}}\right)}{\partial \mathbf{I_{fmax}}} + \lambda_f = 0 \tag{7.54}$$

These conditions are identical to the optimal condition of equation 7.49, except that $\widetilde{\mu}_N$ has been replaced with $\widetilde{\mu}_T$. This is a problem as the two sets of prices are likely to vary considerably. The current injections from the harmonic filter are essentially Norton injections, and hence efficiency requires that they be valued on such a basis. But it is unlikely loads could be charged on the basis of their Norton injections.

The consequence of charging the active filter $\widetilde{\mu}_T$, instead of $\widetilde{\mu}_N$, for harmonic injections is likely to be a sub-optimal level of investment in active filter capacity given that:

$$\mu_T \leq \mu_N \tag{7.55}$$

7.5 PASSIVE FILTER IN A TENSOR ENVIRONMENT

This section starts off initially ignoring the dynamic nature of the nonlinear loads. It was shown in Chapter 6, that the prices which actually represent the true value of harmonic injections when a passive filter was included in the network (equation 6.14), fail to collect the correct amount from the distorting loads. Specifically the amount collected is less than that required to compensate loads for the distortion seen at each busbar, and to pay the filter for injected current. The reason for this being the marginal prices reflect the value of Norton injections into the system, and hence rewarding the injections from a passive filter, using these prices causes problems. This is a similar situation to that described previously in this chapter, where it was shown that nonlinear loads could be charged on the basis of their harmonic injections or Norton injections, but the prices used in each case differed if one was to achieve revenue reconciliation. Of interest is can the techniques that were used to convert the prices for Norton injections into prices for harmonic injections be used to alter the prices in the presence of a passive filter, so that the correct level of harmonic revenue is raised to compensate both the loads and the filter owner?

Previously the injections from the passive filter into the network were specified as:

$$\widetilde{\mathbf{I}}_{\mathbf{f}} = -[Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}} \tag{7.56}$$

This passive filter injection can be respecified in terms of a Norton equivalent:

$$\widetilde{\mathbf{I}}_{\mathbf{f}} = \widetilde{\mathbf{I}}_{\mathbf{Nf}} - [Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_{\mathbf{h}} \tag{7.57}$$

Where the Norton current $\widetilde{\mathbf{I}}_{\mathbf{Nf}}$, is known to be zero.

If the problem of finding the optimal passive filter for the network is internalised, it can be described as:

$$\text{Maximise} \quad U(\mathbf{V_h}) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) \tag{7.58}$$

$$\text{Subject to} \quad \widetilde{\mathbf{I}}_{\mathbf{h}} + \widetilde{\mathbf{I}}_{\mathbf{Nf}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}} = \mathbf{0} \tag{7.59}$$

$$\mathbf{C} \leq \mathbf{C_{max}} \tag{7.60}$$

Note: the ability of loads to reduce their injections is not considered in this example. This has no consequence for the behaviour of interest. The Lagrangian for this problem is:

$$\begin{aligned}\mathcal{L} =& U(\mathbf{V_h}) - PV(\mathbf{C}) - PF(\mathbf{C_{max}}) \\ &+ \widetilde{\mu}_N\left(\widetilde{\mathbf{I}}_{\mathbf{h}} + \widetilde{\mathbf{I}}_{\mathbf{Nf}} - \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)\widetilde{\mathbf{V}}_{\mathbf{h}}\right) + \lambda_P(\mathbf{C_{max}} - \mathbf{C})\end{aligned} \tag{7.61}$$

The first order conditions that describe an optimal allocation of resources are identical to those of Section 6.2.

$$\frac{\partial\mathcal{L}}{\partial\mathbf{V_h}} = \frac{\partial U(\mathbf{V_h})}{\partial\mathbf{V_h}} - \widetilde{\mu}_N\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)[e^{j\theta}] = 0 \tag{7.62}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{C}} = -\frac{\partial PV(\mathbf{C})}{\partial\mathbf{C}} - \widetilde{\mu}_N\frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial\mathbf{C}}\widetilde{\mathbf{V}}_\mathbf{h} - \lambda_P = 0 \tag{7.63}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{C_{max}}} = -\frac{\partial PF(\mathbf{C_{max}})}{\partial\mathbf{C_{max}}} + \lambda_P = 0 \tag{7.64}$$

Equation 7.62 , gives the marginal prices, which were previously specified in Chapter 6 as the true marginal value of harmonic injections into the network.

$$\widetilde{\mu}_N = \frac{\partial U(\mathbf{V_N})}{\partial\mathbf{V_h}}[e^{j\theta}]^{-1}\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \tag{7.65}$$

This price can clearly been seen to represent the marginal value of Norton injections into the network. The price for Norton injections can easily be converted into a price for harmonic injections.

$$\widetilde{\mathbf{I}}_\mathbf{T} = \widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf} - [Y_f(\widetilde{\mathbf{C}})]\widetilde{\mathbf{V}}_\mathbf{h} \tag{7.66}$$

$$\text{Where } \widetilde{\mathbf{V}}_\mathbf{h} = \left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right) \tag{7.67}$$

$$\Rightarrow \widetilde{\mathbf{I}}_\mathbf{T} = \widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf} - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right) \tag{7.68}$$

$$\therefore \frac{\partial\widetilde{\mathbf{I}}_\mathbf{T}}{\partial\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right)} = [I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1} \tag{7.69}$$

Then using the fact:

$$\widetilde{\mu}_N = \frac{\partial\,\text{System Utility}}{\partial\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right)} = \frac{\partial\,\text{System Utility}}{\partial\widetilde{\mathbf{I}}_\mathbf{T}}\frac{\partial\widetilde{\mathbf{I}}_\mathbf{T}}{\partial\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right)} \tag{7.70}$$

The adjusted marginal price for all harmonic injections into the system, including those from nonlinear loads, and passive filters are given by:

$$\widetilde{\mu}_T = \widetilde{\mu}_N\left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h]+[Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)^{-1} \tag{7.71}$$

Using the adjusted marginal prices of equation 7.71, the correct amount is collected from the harmonic injections, i.e.

$$\widetilde{\mu}_N\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right) = \widetilde{\mu}_T\frac{\partial\widetilde{\mathbf{I}}_\mathbf{T}}{\partial\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right)}\left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{Nf}\right) \tag{7.72}$$

$$\frac{\partial U(\mathbf{V_h})}{\partial\mathbf{V_h}}\mathbf{V_h} = \widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{T} \tag{7.73}$$

$$\text{Where } \widetilde{\mathbf{I}}_\mathbf{T} = \widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{f} \tag{7.74}$$

$$\Rightarrow \widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mu}_T\widetilde{\mathbf{I}}_\mathbf{f} = \frac{\partial U(\mathbf{V_h})}{\partial\mathbf{V_h}}\mathbf{V_h} \qquad \text{As required} \tag{7.75}$$

This is an identical result as was achieved using the prices $\widetilde{\mu}_h^s$ of equation 6.41. These two prices can be shown to be identical. From equation 6.75, setting the two prices equal to each other suggests:

$$\begin{aligned}
&\widetilde{\mu}_h^s = \widetilde{\mu}_T \\
&\Rightarrow \left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} = \left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)^{-1} \\
&\left([I] - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\right)\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} = [I] \\
&\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} - [Y_f(\widetilde{\mathbf{C}})]\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)^{-1}\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} = [I] \\
&\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} - [Y_f(\widetilde{\mathbf{C}})][Y_h]^{-1} = [I] \\
&\left([Y_h] + [Y_f(\widetilde{\mathbf{C}})] - [Y_f(\widetilde{\mathbf{C}})]\right)[Y_h]^{-1} = [I] \\
&[I] = [I]
\end{aligned} \tag{7.76}$$

Therefore the process of transforming the marginal prices for Norton injections, into the prices for total harmonic injections, is equivalent to excluding the presence of the filter when calculating the prices. This equivalence will also hold if the dynamic characteristics of the nonlinear loads are included. But only if the variable component of the nonlinear load current is a linear function of the voltage and therefore does not require tensor representation:

$$\text{i.e.} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_1^* \end{bmatrix}$$

$$\text{so that } \widetilde{\mathbf{I}}_{\mathbf{h}} = \widetilde{\mathbf{I}}_{\mathbf{N}} + [Y_1]\widetilde{\mathbf{V}}_{\mathbf{h}}$$

In such cases the price for harmonic injections ($\widetilde{\mu}_T$) is equivalent to the price for Norton injections, if the passive filters and dynamic components of the nonlinear loads are excluded from the network.

Given that $\widetilde{\mu}_T$ and $\widetilde{\mu}_h^s$ are identical the same problems that exists with respect to using $\widetilde{\mu}_h^s$, will exist for $\widetilde{\mu}_T$. Specifically if the filter is charged $\widetilde{\mu}_T$, for their injections, it is optimal for the filter owner to commit filter resources up to the point where:

$$-\widetilde{\mu}_T \frac{\partial[Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} \tag{7.77}$$

The condition in equation 7.77 is clearly different to the optimal condition of equations 7.63 and 7.64. As was found in Chapter 6, there will exist an incentive to over invest in passive filter capacity, as the prices $\widetilde{\mu}_T$, essentially ignore that filter capacity has been added and therefore $\mu_T > \mu_N$.

This problem is similar to that of Section 7.4, where active filters were included in the network and the nonlinear loads were modelled as Norton equivalents. When the marginal prices for the Norton injections were converted to marginal prices for harmonic injections, these prices ($\widetilde{\mu}_T$) failed to provide the correct incentives to the network participants. Except in Section 7.4, it was stated the price for harmonic injections would understate the true value of the injections ($\mu_T < \mu_N$), where here the opposite is being suggested ($\mu_T > \mu_N$). The difference stems from the type of admittance, which is essentially being ignored when $\widetilde{\mu}_T$ is being calculated. The Norton admittance associated with the nonlinear loads aggravates the voltage distortion present, and hence its exclusion from the prices $\widetilde{\mu}_T$, will result in an understatement of the true value of the injections. On the other hand, the passive filter acts to reduce the consequences of any injection, and hence it's exclusion results in the marginal prices $\widetilde{\mu}_T$ overstating the true value of any injections.

Having looked at the inclusion of passive filters, where the nonlinear loads are static current injections, the model is easily extended to consider the inclusion of a passive filter where nonlinear loads need to be represented by tensor Norton equivalents. Initially internalising the filter optimisation problem to characterise the conditions, which describe an efficient allocation of resources, the problem can be stated as:

$$\text{Maximise} \quad \begin{bmatrix} U(\mathbf{V_h}) \\ U^*(\mathbf{V_h}) \end{bmatrix} - \begin{bmatrix} PV(\mathbf{C}) \\ PV^*(\mathbf{C}) \end{bmatrix} - \begin{bmatrix} PF(\mathbf{C_{max}}) \\ PF^*(\mathbf{C_{max}}) \end{bmatrix} \tag{7.78}$$

$$\text{Subject to} \quad \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} - \begin{bmatrix} Y_f(\widetilde{\mathbf{C}}) & 0 \\ 0 & Y_f^*(\widetilde{\mathbf{C}}) \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix}$$

$$- \begin{bmatrix} Y_h & 0 \\ 0 & Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{7.79}$$

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C} \end{bmatrix} \leq \begin{bmatrix} \mathbf{C_{max}} \\ \mathbf{C_{max}} \end{bmatrix} \tag{7.80}$$

In the nodal equation specified above, the Norton current component of the injections from the passive filter have been excluded, as they are equal to zero. The associated Lagrangian for this constrained optimisation problem is:

$$\begin{aligned} \mathcal{L} = & \begin{bmatrix} U(\mathbf{V_h}) \\ U^*(\mathbf{V_h}) \end{bmatrix} - \begin{bmatrix} PV(\mathbf{C}) \\ PV^*(\mathbf{C}) \end{bmatrix} - \begin{bmatrix} PF(\mathbf{C_{max}}) \\ PF^*(\mathbf{C_{max}}) \end{bmatrix} \\ & + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}^*_N \end{bmatrix} \left(\begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{N}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{N}} \end{bmatrix} + \begin{bmatrix} Y_1 - Y_f(\widetilde{\mathbf{C}}) - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_f^*(\widetilde{\mathbf{C}}) - Y_h^* \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} \right) \\ & + \begin{bmatrix} \lambda_f & 0 \\ 0 & \lambda_f^* \end{bmatrix} \left(\begin{bmatrix} \mathbf{C_{max}} \\ \mathbf{C_{max}} \end{bmatrix} - \begin{bmatrix} \mathbf{C} \\ \mathbf{C} \end{bmatrix} \right) \end{aligned} \tag{7.81}$$

The first order conditions, which specify optimality are given in equations 7.82 through 7.84.

$$\frac{\partial \mathcal{L}}{\partial \mathbf{V_h}} = \begin{bmatrix} \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \\ \frac{\partial U^*(\mathbf{V_h})}{\partial \mathbf{V_h}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}^*_N \end{bmatrix} \left(\begin{bmatrix} Y_1 - Y_f(\widetilde{\mathbf{C}}) - Y_h & Y_2 \\ Y_2^* & Y_1^* - Y_f^*(\widetilde{\mathbf{C}}) - Y_h^* \end{bmatrix} \begin{bmatrix} [e^{j\theta}] \\ [e^{-j\theta}] \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.82}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C}} = - \begin{bmatrix} \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} \\ \frac{\partial PV^*(\mathbf{C})}{\partial \mathbf{C}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}_N & 0 \\ 0 & \widetilde{\mu}^*_N \end{bmatrix} \begin{bmatrix} -\frac{\partial Y_f(\widetilde{\mathbf{C}})}{\partial \mathbf{C}} & 0 \\ 0 & -\frac{\partial Y_f^*(\widetilde{\mathbf{C}})}{\partial \mathbf{C}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h}} \end{bmatrix} - \begin{bmatrix} \lambda_f \\ \lambda_f^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.83}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{C_{max}}} = - \begin{bmatrix} \frac{\partial PF(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} \\ \frac{\partial PF^*(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} \end{bmatrix} + \begin{bmatrix} \lambda_f \\ \lambda_f^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.84}$$

From equations 7.83 and 7.84, the optimal level of passive filter cpacity is described by:

$$-\widetilde{\mu}_N \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_{\mathbf{h}} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C_{max}})}{\partial \mathbf{C_{max}}} \tag{7.85}$$

Where the marginal value to the network of Norton injections can be pulled out of equation 7.82, i.e.

$$\widetilde{\mu}_N = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \left(\begin{bmatrix} Y_h + Y_f(\widetilde{\mathbf{C}}) - Y_1 & -Y_2 \end{bmatrix} \begin{bmatrix} [e^{j\theta}] \\ [e^{-j\theta}] \end{bmatrix} \right)^{-1} \tag{7.86}$$

While the prices for the Norton injections represent the true value of injections into the network, the same revenue reconciliation problem still exists. Specifically

$$\widetilde{\mu}_N \left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{f} \right) \neq \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \widetilde{\mathbf{V}}_\mathbf{h} \tag{7.87}$$

Again it is possible to adjust these marginal prices for Norton injections to marginal price for the total injections using equation 7.88.

$$\begin{aligned} \begin{bmatrix} \widetilde{\mu}_T & \widetilde{\mu}_T^* \end{bmatrix} &= \begin{bmatrix} \widetilde{\mu}_N & \widetilde{\mu}_N^* \end{bmatrix} \\ &\times \left([I] + \begin{bmatrix} Y_1 - Y_f(\widetilde{\mathbf{C}}) & Y_2 \\ Y_2^* & Y_1^* - Y_f^*(\widetilde{\mathbf{C}}) \end{bmatrix} \begin{bmatrix} Y_h + Y_f(\widetilde{\mathbf{C}}) - Y_1 & -Y_2 \\ -Y_2^* & Y_h^* + Y_f^*(\widetilde{\mathbf{C}}) - Y_1^* \end{bmatrix}^{-1} \right)^{-1} \end{aligned} \tag{7.88}$$

Where both a passive filter and dynamic nonlinear loads are included, it is not possible to make any general statements as to the magnitude of the prices $\widetilde{\mu}_T$, compared with the magnitude of the price for the Norton injections $\widetilde{\mu}_N$. The passive filter will act to make $\mu_T < \mu_N$, while the opposite is true of the Norton admittance associated with the nonlinear loads. As before this adjustment of the marginal prices will result in the harmonic payments from the loads being equal to the compensation payments to each load, along with the amount owing to the filter for their injections.

$$\widetilde{\mu}_T \left(\widetilde{\mathbf{I}}_\mathbf{h} + \widetilde{\mathbf{I}}_\mathbf{f} \right) = \frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}} \widetilde{\mathbf{V}}_\mathbf{h} \tag{7.89}$$

But while the marginal prices $\widetilde{\mu}_T$, will collect the correct amount from all those making injections into the network, as was previously the case, these prices will fail to provide the correct incentive to the potential filter owners. It will optimal for the filter owner to invest up to the point where:

$$-\widetilde{\mu}_T \frac{\partial [Y_f(\widetilde{\mathbf{C}})]}{\partial \mathbf{C}} \widetilde{\mathbf{V}}_\mathbf{h} = \frac{\partial PV(\mathbf{C})}{\partial \mathbf{C}} + \frac{\partial PF(\mathbf{C}_{\mathbf{max}})}{\partial \mathbf{C}_{\mathbf{max}}} \tag{7.90}$$

Clearly the condition in equation 7.90 differs from the network optimum shown in equation 7.85.

7.6 CONCLUSION

The harmonic injections of nonlinear loads are not those of a static current source. These injections will almost certainly be dependent on the voltage seen at the local busbar. In the theoretical case of an infinitely strong system, the voltage at each busbar consists only of fundamental irrespective of what the harmonics the loads inject. Under such conditions the harmonic injections could validly be modelled as static, but as harmonic distortion cannot exist, there is no need to value the injections. The point being, so long as there is some voltage distortion throughout the system, there is a need to consider how this distortion affects the injections of each of the nonlinear loads.

With marginal pricing, of interest is how the load behaves at the margin, as a result there is no difference between modelling the load as a voltage dependent current source, or as a Norton equivalent. Modelling the nonlinear loads as Norton equivalents is convenient on account of the fact it helps the interpretation of the prices, and identification of what is 'of value'. Specifically if the dynamic characteristics of the load is to be included in the pricing model, what the marginal prices actually represent is the marginal value of any Norton injections into the network. This is potentially a problem as the Norton injections are a mathematical construct. As a consequence it is impossible to verify what the Norton injections for each load are with out explicitly modelling each load in the network. In the case where the harmonic injections of the nonlinear loads are close to static, the total injections will be close to the constructed Norton injections. Under such circumstances using the total injections as a proxy for the Norton injections will not be of much consequence.

Unfortunately the it is not possible to impose such a constraint on the physical system, and demand that all nonlinear loads behave as static or near static current sources. As was shown, it is possible to transform the prices for Norton injections ($\widetilde{\mu}_N$) into prices for the total harmonic injections ($\widetilde{\mu}_T$). The adjusted prices for total harmonic injections have both advantages and disadvantages. The main advantage is the charges for each load are based on their total harmonic injections ($\widetilde{\mathbf{I}}_{\mathrm{h}}$). If the prices for Norton injections are used, presumably loads must be charged on the basis of their Norton injections ($\widetilde{\mathbf{I}}_{\mathrm{N}}$). The fact calculation of the Norton prices and injections is difficult (read practically impossible), has been mentioned, but of more importance, is the fact that the harmonic injections are the real tangible cause of all the distortion throughout the network. Looking at the example of Section 7.3, the majority of the voltage distortion was caused by the load at busbar four, which had the variable Norton impedance. Most would agree, as that load was responsible for the majority of the distortion, it should bear the majority of the compensation costs. If one charges loads on the basis of their harmonic injections this is what occurs. On the other hand, if the loads are charged on the basis of their Norton injections, the load at busbar four is charged an amount comparable to the other loads, this is clearly not reasonable.

The disadvantage of using the adjusted prices for harmonic injections ($\widetilde{\mu}_T$) is that they fail to provide the correct incentives to network participants, i.e. aggregate network and private welfare diverge. The price for Norton injections ($\widetilde{\mu}_N$) do represent the true marginal cost to the network of any additional harmonic injections into the network. When making decisions about committing resources to harmonic mitigation, for these decisions to prove optimal the true value of any mitigation action taken must be known. Clearly if the harmonic prices ($\widetilde{\mu}_T$) are used, these fail to articulate what is the true value of any injection.

Of interest is the equivalence between the prices for harmonic injections, and the Norton prices where the dynamic qualities of the loads, and any passive filters are ignored. This intuitively makes sense in that if one considers the total injection of the nonlinear load as a single quantity, this is an injection into a system that excludes the Norton admittance of all the loads.

The question is, what set of prices should be used? To this there is no clear answer. Charging loads on the basis of their Norton injections is not an attractive offer for a number of reasons. It is also worth considering that if harmonic charges are to be based on Norton injections, this results in no value being attributed to the injections of a passive filter. Given the potential importance passive filters may play in the mitigation of voltage distortion, the pricing system must reward passive filters. Passive filter could be rewarded using some criteria outside of the pricing system, but this would defeat the purpose. Charging loads $\widetilde{\mu}_T$ for their harmonic injections cannot achieve an efficient outcome, especially in the presence of passive filters. The degree to which this will fail to achieve optimality will depend on the system. But while not perfect, using the prices $\widetilde{\mu}_T$, and charging loads on the basis of harmonic injections is feasible and as shown, the correct amount will be collected so that the books balance. Another option is to use the prices for Norton injections $\widetilde{\mu}_N$, but apply it to the total harmonic injections of each load. This is attractive in that it will encourage efficient behaviour, and hence aggregate network welfare will be maximised. However the books will not balance and there may potentially be large cash flows into or out of the system to finance the difference between what comes in from the harmonic charges and what goes out the form of payments to filter owners and compensation for voltage distortion. The best solution to this problem is likely to depend on circumstance.

Chapter 8

CONCLUSIONS AND FUTURE WORK

8.1 CONCLUSIONS

"The economic problem of society is thus not merely a problem of how to allocate 'given' resources,...rather a problem of how to secure the best use of resources known to any members of society, for ends whose relative importance only these individuals know. It is a problem of utilization of knowledge not given to anyone in its totality." [Hayek1945]

It has been shown that harmonic distortion reduces the welfare of loads in the network. Simply put harmonic distortion may prevent proper operation of equipment, and cause accelerated wear or destruction. The consequences of harmonic distortion are known and individuals go to considerable, and often expensive lengths to minimise the potential damage which can result. The goal of any harmonic mitigation action is to efficiently commit resources to the problem. In other words when solving the problem, every dollar spent at the margin should improve the welfare of the aggregate network by an equivalent amount.

Given a full information set this constrained optimisation problem is easily solved. Any individual, with knowledge as to the preferences of each individual in the network, the technology available and circumstances of each individual along with the all the physical specifications of the network, can easily use this information to determine the optimal allocation of resources. This allocation will be Pareto efficient. In each chapter a similar process was undertaken to characterise the optimal allocation of resources that is desirable. Clearly the weakness in the above approach is that no single entity has this complete knowledge set. A network operator may well have a very good idea as to the technology available and the physical specifications of the network, but they certainly will not have much idea as to the preferences of each of the loads. As it turns out it is these preferences that are the single most important piece of information, as they determine the severity of the harmonic problem experienced and what loads will give in return for reducing the harmonic levels.

How effective are the standards and regulations used in most networks in utilising the information dispersed throughout the network? The answer is clearly not very effective, on account of the fact the standard and regulations make no effort to discover the preferences of the loads, instead they implicitly assume that each load has identical preferences of a particular form. To their credit the standards are often crafted with an eye towards the requirements of modern technology and the harmonic mitigation technology available. But none-the-less, it is clear given their inability to harvest and use information about individuals preferences, standards mandated by the network operator can never come close to achieving theoretical optimal allocation of resources.

The other option to a central authority dictating the required behaviour of each load, is to decentralise the decision making process. As before, the question is how well is information used with a decentralised decision making process? Clearly the one advantage is each load knows its own preferences, then given information as to the value of harmonic injections the loads are in a position to make rational decisions as to allocating resources. An efficient allocation will occur so long as loads know the value of the harmonic injections, however they do not need to be aware the underlying determinants of the value. The marginal pricing techniques developed in this work represent an attempt to develop prices that accurately reflect the value of harmonic injections, and hence can be used by individual loads to make decisions, which while independent of each other, collectively act to maximise aggregate welfare. The prices if effective, will essentially be a summary of every thing the individual loads need to know to act in an efficient manner.

The question is, "how well do the marginal prices make use of all the dispersed information, and encourage the separate loads to behave in an efficient manner?" It was found in the case where there are no filters in the network and nonlinear loads behave as static current sources, by acquiring information as to the preferences of each load, and using information about the physical parameters of the network, it is possible to develop marginal prices that accurately reflect the true value of injections throughout the network and which encourage efficient behaviour by each load. How will this information be acquired to incorporate into the prices developed, and will this information be accurate? The marginal pricing system as put forward here simply extracted a value for harmonic distortion from each load. Unfortunately it was found that it might potentially be difficult to extract this information from the loads, as they might have incentives to mislead others about their valuation of distortion. Depending on whether the loads have the right to a clean supply or the right to inject what they wish into the network, it was demonstrated incentives potentially exist for loads to over or understate their harmonic valuations. In the case where each load is small compared to the network no such incentive exists. Therefore marginal pricing, while not perfect at extracting and making use of individual preferences, works reasonably well. Also for the marginal prices to accurately reflect the true value of the injections they require information as to the physical state of the network. It is here that some central authority comes into play. At present all the required information with respect to the harmonic voltages throughout the network are unavailable. Also information as to what the network admittance looks like, is not known by each individual. This information, if available, is likely to only be known by the system operator. Hence marginal pricing requires that the system operator acts to collect (as best possible) information as to the individual preference and then incorporates the knowledge it has with respect to the physical state of the system, to develop the final prices.

Marginal pricing may be unable to extract the true valuation some loads place on the voltage distortion seen at their busbar. While a shortcoming, this does not consign marginal pricing to being of little value. While the loads are 'small' there exists no incentive for the loads to mis-state their valuation of harmonic distortion. But more importantly, there have been no better alternatives put forward for making use of this information, as the standards in place disregard that each of the loads is likely to have different preferences. One possibility is that for large loads which have an incentive to mis-state their harmonic valuation, it may be desirable to have the central authority that collates the prices, to do an audit to ensure that the harmonic valuation given is reasonable. With such a system in place the marginal pricing system, as put forward would do an exceptional job of extracting and communicating the required information to each load that encourages efficient allocation of resources.

As discussed in Chapter 6, extra problems creep in when considering the possibility of passive filters being included in the network. The owner of a passive filter potentially acquires a large amount of market power. At this point the owners actions change from responding to the price signals to manipulating the prices for maximum gain. This has negative consequences for the aggregate system, as the marginal prices if distorted cannot provide the correct incentives to all the other loads in the network. Given the important role passive filters potentially play in dealing with distortion, marginal pricing will be a failure if it is unable to encourage an efficient use of filter resources. It was found by ensuring that no filter capacity is withheld from the network, marginal pricing could work to efficiently allocate filter resources. In fact marginal pricing has the potential to considerably improve the current allocation of filter resources, as it should ensure the preferences of the loads are considered, and also allow the knowledge of every individual in the network to be put to use. Even with marginal pricing it is likely the majority of the filters included in network will be installed by the network owner/operator. But by allowing everyone to potentially profit from an investment in a filter which adds value, marginal pricing effectively harnesses the collective knowledge of the total network.

The above point raises an important issue, in that this work does not suggest that there is no place for some form of system operator making decisions with respect to the allocation of resources to mitigate the costs of harmonic injections. To the contrary, on account of the fact each load has no way of knowing what proportion of the distortion seen at their busbar is attributable to each load in the network, the system operator is required to calculate the marginal prices. The system operator is a participant in the electrical network with considerable expertise in knowledge in the analysis and operation of filters. Therefore to exclude the system operator from building and profiting from filters is illogical. It is suggested that the basis of their decisions should be the marginal prices, on account of the fact they make superior use of the available knowledge, specifically the preferences of each load in the network.

The prices for harmonic injections will be complex and in general the amount charged to each load will also be complex. As detailed in Chapter 4, should only the real part of the harmonic charges due from any load are of interest. This is important as it makes charging for harmonic injections viable. Clearly if loads had to pay some complex amount for their injections, implementation of marginal pricing would be impossible. The complex prices and amounts calculated are a consequence of the fact any injection into the network will have in phase and quadrature components, with respect to the prevailing voltage at their busbar. The complex amount charged to any load represents the value of the quadrature component of their harmonic injections, had it been in phase with the prevailing voltage. But as the quadrature component of all injections has no effect on the voltage magnitude at the margin, it need not be considered for marginal pricing purposes.

The final problem associated with marginal pricing relates to how the market should be structured. It was shown at a fundamental level, the quantities of value are the Norton injections made into the network. The price associated with these injections indicate what is the true marginal cost of any current injection into the network. Calculating the price for Norton injections requires models of every nonlinear load in the network. Should it be possible to calculate this price, using it to charge for the total harmonic injections of each load will fail to collect the required revenue. On the other hand prices can be calculated for the total harmonic injections that are based on the premise the nonlinear loads and filters are not part of the network per se. These prices fail to reflect the fact that the nonlinear loads and the filters do effect the value of any harmonic injection into the network. There would seem to be no neat solution to this problem.

As demonstrated, marginal pricing is attractive as it potentially summarises all the information relating to the physical state of the network and the preferences of each load, so that each individual is encouraged to act in the best interests of the system. The potential impediments to marginal pricing being efficient is that individuals must reveal their personal preferences honestly, and in some circumstances excess market power will result in a divergence between the collective and individual utility. The other final potential impediment is there is a potentially massive amount of information, which must be incorporated into the prices if they are to encourage optimal behaviour from each load. Much of this information is unavailable at present.

8.1.1 Achievement Summary

Briefly summarised the main achievements of this work are:

- The required information to achieve an optimal allocation of harmonic resources is specified.
- The specification of the optimal marginal prices under different network models and conditions ranging from the simplest network models, to networks including passive and active filters, and systems where the nonlinear loads are modelled as voltage dependent current sources, have been achieved.
- Demonstration that given the required information, marginal prices can be formed which will encourage individual loads to behaviour efficiently. The marginal prices align individual and aggregate network welfare.
- Specification of the conditions that are likely to cause market failure, and consequences of any such failure.
- An analysis of the resultant amounts charged/paid to different types of loads and filters using marginal pricing. Demonstrating that while marginal pricing will specify complex payments from each load, only the real part of such payments need to be considered for marginal pricing to be efficient.
- Establishment that the fundamental elements of value are the Norton injections into the network.
- Techniques to impliment marginal pricing in the absence of any measured or calculated Norton injection values.
- Investigation of the formulation of a harmonic market, including specification of some of the important issues associated with such a market.

8.2 FUTURE WORK

This work has developed methods to make use of all the possible information, which will result in an optimal allocation of harmonic resources. But naturally the methods developed can only be used in the case where all the required information is available. It will be quickly apparent that at present there is a serve case of information scarcity, in which the required information to implement marginal pricing as postulated, does not rest with any member of the network. A good economic system is one, which fully utilizes all information, but any system cannot be judged on its failure to utilize information that does not exist. To that extent comparing any system for allocating harmonic resources to the marginal prices developed in this work, and declaring them inefficient is not valid, on the basis that the marginal harmonic prices are not a feasible allocation mechanism at present, because of the information scarcity that exists. This would seem to fly in the face of the previous criticism of the use of standards to allocate harmonic resources. Certainly one cannot declare that standards are presently an inferior allocation mechanism compared to marginal harmonic pricing, on account of the fact standards are implementable, while the marginal prices are not. But the criticism that standards make no attempt to utilize all the information presently available, and will fail to incorporate new information as technology makes it available is valid.

8.2.1 Information Scarcity

There is a need to investigate the extent to which information scarcity exists, with respect to the implementation of marginal pricing. There are two different sets of prices developed in this work, the marginal prices for total harmonic injections (including filters) $\tilde{\mu}_T$, and the marginal prices for Norton injections $\tilde{\mu}_N$. The two sets of prices require different amounts of information and as such their implementation is impeded by information scarcity to different extents.

Table 8.1 Required information for different marginal forms of marginal pricing

	Available Information	**Scarce Information**
Harmonic	Individual Preferences $\frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}$	Harmonic Injection Magnitude $\mathbf{I_h}$
Prices	Basic Admittance Matrix $[Y_h]$	Harmonic Injection Angle $\boldsymbol{\alpha}_h$
$\tilde{\mu}_T$	Harmonic Voltage Magnitude $\mathbf{V_h}$	Harmonic Voltage Angle $\boldsymbol{\theta}_h$
Norton	Individual Preferences $\frac{\partial U(\mathbf{V_h})}{\partial \mathbf{V_h}}$	Harmonic Injection Magnitude $\mathbf{I_h}$
Prices	Basic Admittance Matrix $[Y_h]$	Harmonic Injection Angle $\boldsymbol{\alpha}_h$
$\tilde{\mu}_N$	Harmonic Voltage Magnitude $\mathbf{V_h}$	Harmonic Voltage Angle $\boldsymbol{\theta}_h$
	Filter Admittance Matrix $[Y_f(\tilde{\mathbf{C}})]$	Norton Admittance of Nonlinear Loads $\begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix}$

Note that the distinction between what information is available and scarce as detailed in Table 8.1, will not hold for every network. Certainly in some cases information as to the harmonic voltages throughout the network will not be available, while in other cases information as to the harmonic injections of loads might be available. Table 8.1, is only indicative, and is to an extent based on the difficulty of gathering the information. Of interest is that for any network where harmonic injection information is not collected, enforcing a standard such as IEEE 519 is not possible. In such cases a different resource allocation mechanism will be required, as implementing a standard to which compliance cannot be tested serves no purpose.

8.2.2 Immediate Solution

One area of future work is to consider, given what information presently exists throughout the network, what is the best possible resource allocation mechanism possible. Given that each load knows about their own preferences, and the system operator might have some idea as to what the admittance matrix looks like, and what are the harmonic voltages around the network; what economic system will allow an allocation of resources, which is as close as possible to the optimal allocation, associated with complete knowledge. Naturally a big part of this problem is developing systems so that loads honestly reveal their preferences. Given the very limited information available in most networks some sort of standard may be the best option, but only if compliance with the standard can be determined, and if the standard incorporates the preferences of each load.

8.2.3 Information Accumulation

The main impediment to achieving an optimal allocation of resources is information scarcity. It must be established exactly what information is of value, and how this information can be accumulated. But in doing this the potential benefits associated with the accumulation of the information must be considered i.e. an attempt must be made to calculate the expected marginal increase in system welfare that accrues due the accumulation of all types of knowledge, and compare it to the marginal cost of acquisition. Many types of information while potentially useful, are no doubt prohibitively expense to acquire, this problem should ease with the advancement of metering technology.

8.2.4 Full Harmonic Solution

This work only considered a single harmonic order. The techniques should be extended to include all harmonic orders of interest. A useful advancement would be to move to a full harmonic domain solution where the coupling between all the harmonic orders would be captured. Also with multiple harmonic orders, consideration must be made of the structure of the harmonic utility functions. Not only will loads have preferences with respect to the magnitude of each harmonic at their busbar, but they are also likely to have preferences linked to the aggregate distortion, which is represented by measures such as total harmonic distortion. Of interest is how utility functions liked to multiple harmonic frequencies and THD affect the valuation of different types of injections.

This will also present a new set of difficulties in that information will be required as to the relative values loads place on different harmonic orders. The way different harmonic orders interact will have different implications for each load and this too will need to be accounted for the utility function for each load. This comes back to the question of just what value do different types of loads loads place on different types of harmonic distortion? A question that has yet to be answered.

8.2.5 Three Phase Solution

This work assumed balanced loads. An improvement will result from the to move to a three phase solution, so that the consequences of load imbalance can be considered. Load imbalance can play an important role in determining the harmonic state of the network, given that negative sequence fundamental can be a major cause of harmonic injections from power electronic converters. The consideration of unbalance in the solution will require the incorporation of fundamental power flow into the pricing system.

8.2.6 Harmonic Current

This work was based on the utility of each load being a function of the harmonic voltage distortion at their busbar. The harmonic current flows also have costs associated with them. The owner of the transmission and distribution assets almost certainly would like to see the harmonic current flows across their networks minimised, to ensure correct operation and avoid the premature aging of the assets. For the harmonic allocation mechanism to be efficient, the costs associated with both harmonic currents and voltages must be considered. Also harmonic sources that may exist in the network such as the nonlinear magnetising characteristics of transformers must be considered.

Appendix A

TEST SYSTEM

The following appendix details the base case test system, which is used throughout the thesis. Figure A.1 details the layout of the nine busbar test system, and indicates the length of the of the transmission lines that connect each busbar. The impedance of the transmission lines is $1.719 \times 10^{-3} + j4.402 \times 10^{-3} pu/mi$, at fundamental.

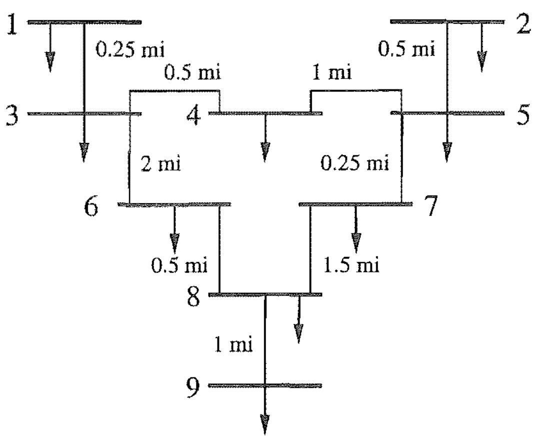


Figure A.1 Schematic of the nine busbar test system

There is assumed to be only one load at each busbar. Each load is part linear and part nonlinear. Details of the linear and nonlinear load at each busbar is given in Table A.1. The table also indicates the fifth harmonic current injected by each load in the simplest case where all injections are made at a common angle. The injected harmonic current is that which would result for six-pulse converter in the presence of a undistorted commutating voltage, where the firing angle is 15° [Arrillaga1983]. For simplicity, through out the thesis only the fifth harmonic is considered. The techniques are easily extended to allow consideration of multiple harmonic orders.

Table A.2 details the (base case) valuation each load places on voltage distortion seen at their busbar, where this is a linear function. Also contained is the marginal cost of a reduction in injected harmonic current, for each load, where this is a constant amount per unit.

In many of the examples certain elements of the above test system of varied. In such examples, elements that are not explicitly stated as a variable will have a value as detailed in this appendix.

Table A.1 Test System Load Data

Bus	Linear Load (pu)	Nonlinear Load (pu)	Injected 5^{th} Harmonic Current (pu)
1	$0.58 + j0.16$	$0.29 + j0.08$	$0.060e^{j\frac{7}{12}\pi}$
2	$2.00 + j0.00$	$0.04 + j0.01$	$0.008e^{j\frac{7}{12}\pi}$
3	$0.58 + j0.16$	$0.29 + j0.08$	$0.060e^{j\frac{7}{12}\pi}$
4	$1.16 + j0.31$	$1.45 + j0.39$	$0.300e^{j\frac{7}{12}\pi}$
5	$2.00 + j0.00$	$0.04 + j0.01$	$0.008e^{j\frac{7}{12}\pi}$
6	$1.16 + j0.31$	$1.45 + j0.39$	$0.300e^{j\frac{7}{12}\pi}$
7	$0.38 + j0.10$	$0.19 + j0.05$	$0.040e^{j\frac{7}{12}\pi}$
8	$0.29 + j0.08$	$0.14 + j0.04$	$0.024e^{j\frac{7}{12}\pi}$
9	$0.38 + j0.10$	$0.19 + j0.05$	$0.040e^{j\frac{7}{12}\pi}$

Table A.2 Test System Valuation Data

Bus	Harmonic Valuation $\mathbb{K}$ (\$/pu)	Harmonic Reduction Cost ρ (\$/pu)
1	-60	60
2	-10	100
3	-50	25
4	-70	80
5	-10	100
6	-50	15
7	-30	42
8	-40	58
9	-50	42

Appendix B

PUBLIC GOOD

It was stated that the harmonic current injections have properties similar to that of a public good. Hence when loads are given the right to pollute, the lack of an organised harmonic market might result in a sub-optimal resource allocation. To prove this examine the decision making process for a single load at busbar t (referred to as t). Load t, will be willing to pay any other load at a busbar s (referred to as s), to reduce their harmonic injections, an amount up to t's marginal valuation of the injected harmonic current by s. As such t, is willing to pay s, an amount up to:

$$\left(\frac{\partial u_t}{\partial V_{ht}}\right) y_{ts}^{-1} \tag{B.1}$$

Which in the case of constant marginal utility equals $k_t y_{ts}^{-1}$. Load t, will be willing to pay the load at busbar s, to reduce their harmonic injections should the condition described in equation B.2 exist.

$$\left(\frac{\partial u_t}{\partial V_{ht}}\right) y_{ts}^{-1} < -\frac{\partial RC_s(I_{Rs})}{\partial I_{Rs}} \tag{B.2}$$

To simplify the notation and graphical presentation of this appendix it is assumed that cost to load s, of harmonic injection reduction, is a linear function of reduction magnitude ie. constant marginal cost of harmonic current reduction.

$$\frac{\partial RC_s(I_{Rs})}{\partial I_{Rs}} = \rho_s \tag{B.3}$$

This assumption does not change any of the important results. Using this assumption equation B.2 can be restated as:

$$\left(\frac{\partial u_t}{\partial V_{ht}}\right) y_{ts}^{-1} < -\rho_s \tag{B.4}$$

Despite that the combined benefit to the whole network (as given in the shadow price $\mu_{hs} = \sum_i^n k_i y_{is}^{-1}$) may exceed the cost of the reduction in current, no one load will be willing to bear this whole cost as $\|k_t y_{ts}^{-1}\| < \|\mu_{hs}\| \quad \forall t$. As a consequence the good (harmonic reduction) will not be provided despite the fact it efficient to do so.

For example if one is to look at the decision making process for a single load t. Here t is considering whether to pay s to reduce their harmonic injections. Load t has an endowment of

(g_{all}, w_t), where

$$\begin{array}{lcl} g_{all} & \rightarrow & \text{Amount of harmonic reduction purchased by} \\ & & \text{all other network participants} \\ g_t & \rightarrow & \text{Amount of harmonic reduction purchased by } t \\ w_t & \rightarrow & \text{Monetary wealth of } t \end{array}$$

If s has a constant marginal cost for harmonic reduction ρ_s.

$$\begin{array}{rcl} \text{Current Reduction } G & = & g_t + g_{all} \\ \text{Private Consumption } x_t & = & w_t - \rho_s g_t \end{array}$$

As such when deciding how much to pay for a reduction in injected harmonic current, load t, faces a constrained optimisation problem:

$$\max \quad u_t(G, x_t) = u_t(g_t + g_{all}, w_t - \rho_s g_t)$$

Subject to the constraints

$$g_t \geq 0 \tag{B.5}$$

$$g_t + g_{all} \leq I_{hs} \tag{B.6}$$

This problem has the following equilibrium condition

$$\begin{aligned} \frac{\partial u_t}{\partial G} &\leq \lambda + \rho_s \frac{\partial u_t}{\partial x_t} \\ \frac{\frac{\partial u_t}{\partial G}}{\frac{\partial u_t}{\partial x_t}} &\leq \frac{\lambda}{\frac{\partial u_t}{\partial x_t}} + \rho_s \end{aligned} \tag{B.7}$$

If we assume that constraint B.6, is not binding ($\Rightarrow \lambda = 0$), then

$$\frac{\frac{\partial u_t}{\partial G}}{\frac{\partial u_t}{\partial x_t}} \leq \rho_s \tag{B.8}$$

Equation B.8 indicates in equilibrium, load t will have a marginal rate of substitution between private consumption and harmonic reduction, that is less than or equal to the price ratio of the two goods ρ_s, (as the price of private consumption is defined as one). Optimality demands they be equal. In the case where the two ratios are equal, we have a situation like that shown in Fig. B.1. In this case the indifference curve of the load at busbar bar t, is tangent to the budget line. (Note the slope of the budget line will be $1/\rho_s$). Hence in the case where the load's preferences are such that they will want to contribute to the public good we will still have optimal result where the ratio of marginal benefits equals the ratio of marginal costs.

But the above situation will not occur for each load. It can be shown, should provision of a public good be left to private decision, only the individual with the highest valuation of the public good will contribute. Once the load with the highest valuation of the public good has finished contributing, the marginal cost of further reduction in the harmonic injections will exceed the private valuation of any other load. As such it will be optimal for every other individual load to choose to contribute nothing to the public good despite that it would be collectively optimal to do so. This incentive for an individual to free ride is shown in Fig. B.2.

There is a limit as to how much of the public good can be provided. As stated in the constraints, the harmonic input of s can only be reduced to zero. There may be occasions where a load t, would like to see s sink the harmonic current from the system. In this situation where

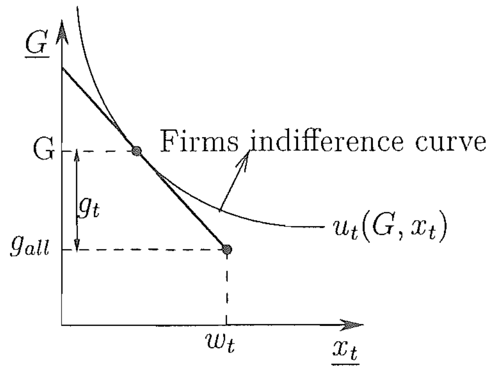


Figure B.1 The indifference curve and budget set where load t, chooses to contribute to the reduction of harmonics injected by s

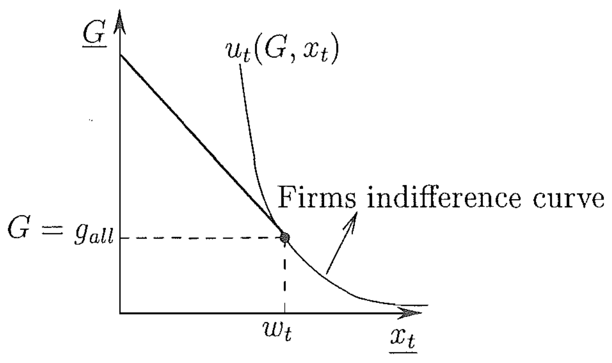


Figure B.2 The indifference curve and budget set where t, chooses to free ride on other loads contributions

the constraint is binding:

$$\frac{\frac{\partial u_t}{\partial G}}{\frac{\partial u_t}{\partial x_t}} \leq \frac{\lambda}{\frac{\partial u_t}{\partial x_t}} + \rho_s > \rho_s \tag{B.9}$$

This is represented graphically in Fig. B.3

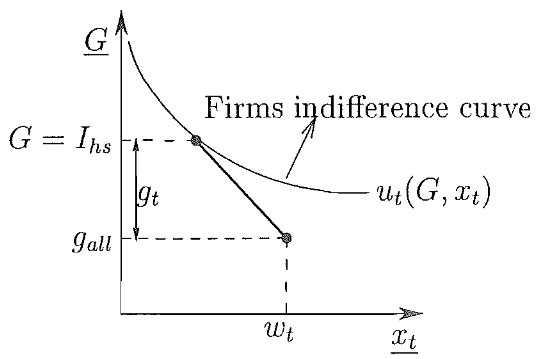


Figure B.3 The indifference curve and budget set for t, where the constraint is binding

It is of interest to compare equation B.7 (the first order conditions for an individual decision making process) with equation 3.31 (the first order conditions for harmonic system optimisation). Rewriting equation 3.31 to reflect the linear harmonic reduction costs assumed, the result is:

$$-\rho_s = \mu_{hs} - \lambda_{1s} + \lambda_{2s} \tag{B.10}$$

In developing equation B.10, there was a tacit assumption that the marginal utility of private consumption is 1, for all loads. This is as the problem was constructed as:

max Utility(Voltage Harmonics) - Cost(Harmonic Reduction)

Using the definition of the shadow price μ_{hs} one can find the ratio of, system marginal utility of harmonic reduction, to the marginal utility of private consumption (by definition equal to 1).

$$\mu_{hs} = \frac{\partial U}{\partial I_{hs}} = -\frac{\partial U}{\partial G} = \frac{-\frac{\partial U}{\partial G}}{\frac{\partial U}{\partial x}} \tag{B.11}$$

Combining B.10 and B.11, one ends up with an expression identical in form to B.7

$$\frac{\frac{\partial U}{\partial G}}{\frac{\partial U}{\partial x}} = \rho_s - \lambda_{1s} + \lambda_{2s} \tag{B.12}$$

One can also develop this expression by solving the problem

$$\begin{array}{rc} & \max \quad U(\mathbf{V_h}, x) \\ & \text{subject to} \\ & -\mathbf{I_{hr}} \leq \mathbf{0} \\ & \mathbf{I_{hr}} \leq \mathbf{I_{h*}} \\ & \mathbf{I_{h*}} = \mathbf{I_{hr}} + [Y_h]\widetilde{\mathbf{V}}_{\mathbf{h}} \\ \text{Where} & x = \text{private consumption} = w - \boldsymbol{\rho}\mathbf{I_{hr}} \end{array}$$

Equation B.12 is identical to B.7, expect that u_t has been replaced with U. This shows that the conditions which result in overall system optimisation, do not coincide with the outcome that results from a private decision making process, where there is a lack of an organised market and should property rights be allocated so that loads have a right to pollute. A background to public good economics can be found in [Varian1992].

Appendix C

BUSBAR VOLTAGE MAGNITUDE

Here it is demonstarted how equation 4.6 is derived. The voltage at each busbar is given by C.1.

$$\widetilde{\mathbf{V}}_{\mathbf{h}} = \begin{pmatrix} \gamma_1 e^{j(\beta^{-1}+\alpha)} + \xi_1 e^{j(\beta^{-1}+\alpha_z)} \\ \gamma_2 e^{j(\beta^{-1}+\alpha)} + \xi_2 e^{j(\beta^{-1}+\alpha_z)} \\ \vdots \\ \gamma_n e^{j(\beta^{-1}+\alpha)} + \xi_n e^{j(\beta^{-1}+\alpha_z)} \end{pmatrix} \tag{C.1}$$

Taking the magnitude of each row of the above vector

$$\mathbf{V}_{\mathbf{h}} = \begin{pmatrix} [\left(\gamma_1 \sin(\beta^{-1}+\alpha) + \xi_1 \sin(\beta^{-1}+\alpha_z)\right)^2 + \left(\gamma_1 \cos(\beta^{-1}+\alpha) + \xi_1 \cos(\beta^{-1}+\alpha_z)\right)^2]^{\frac{1}{2}} \\ [\left(\gamma_2 \sin(\beta^{-1}+\alpha) + \xi_2 \sin(\beta^{-1}+\alpha_z)\right)^2 + \left(\gamma_2 \cos(\beta^{-1}+\alpha) + \xi_2 \cos(\beta^{-1}+\alpha_z)\right)^2]^{\frac{1}{2}} \\ \vdots \\ [\left(\gamma_n \sin(\beta^{-1}+\alpha) + \xi_n \sin(\beta^{-1}+\alpha_z)\right)^2 + \left(\gamma_n \cos(\beta^{-1}+\alpha) + \xi_n \cos(\beta^{-1}+\alpha_z)\right)^2]^{\frac{1}{2}} \end{pmatrix} \tag{C.2}$$

Looking at the first element of the vector in C.2

$$\begin{aligned} \mathbf{V}_{\mathbf{h}}(1,1) &= [\left(\gamma_1 \sin(\beta^{-1}+\alpha) + \xi_1 \sin(\beta^{-1}+\alpha_z)\right)^2 + \left(\gamma_1 \cos(\beta^{-1}+\alpha) + \xi_1 \cos(\beta^{-1}+\alpha_z)\right)^2]^{\frac{1}{2}} \\ &= [\gamma_1^2 \sin^2(\beta^{-1}+\alpha) + 2\gamma_1\xi_1 \sin(\beta^{-1}+\alpha)\sin(\beta^{-1}+\alpha_z) + \xi_1^2 \sin^2(\beta^{-1}+\alpha_z) \\ &\qquad + \gamma_1^2 \cos^2(\beta^{-1}+\alpha) + 2\gamma_1\xi_1 \cos(\beta^{-1}+\alpha)\sin(\beta^{-1}+\alpha_z) + \xi_1^2 \cos^2(\beta^{-1}+\alpha_z)]^{\frac{1}{2}} \\ &= [\gamma_1^2 + 2\gamma_1\xi_1 \left(\sin(\beta^{-1}+\alpha)\sin(\beta^{-1}+\alpha_z) + \cos(\beta^{-1}+\alpha)\cos(\beta^{-1}+\alpha_z)\right) + \xi_1^2]^{\frac{1}{2}} \\ &= [\gamma_1^2 + \gamma_1\xi_1 \left(\cos(\alpha-\alpha_z) - \cos(2\beta^{-1}+\alpha+\alpha_z) + \cos(\alpha-\alpha_z) + \cos(2\beta^{-1}+\alpha+\alpha_z)\right) + \xi_1^2]^{\frac{1}{2}} \\ &= [\gamma_1^2 + 2\gamma_1\xi_1 \cos(\alpha-\alpha_z) + \xi_1^2]^{\frac{1}{2}} \end{aligned} \tag{C.3}$$

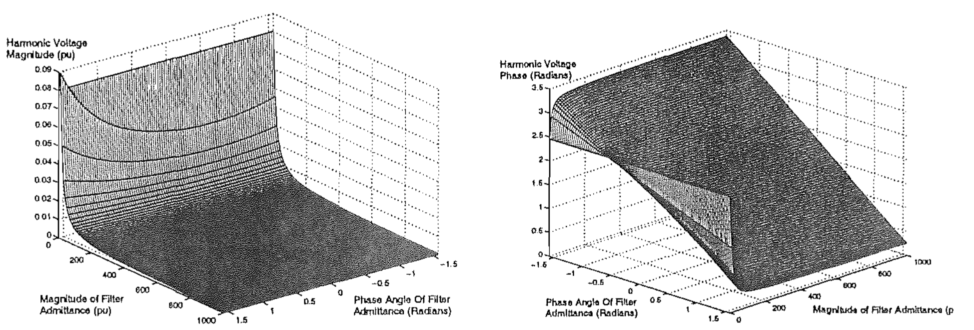
Equation C.3, can then be generalised to each busbar throughout the network to produce the desired result.

$$\mathbf{V}_{\mathbf{h}} = \begin{pmatrix} [\gamma_1^2 + 2\gamma_1\xi_1 \cos(\alpha-\alpha_z) + \xi_1^2]^{\frac{1}{2}} \\ [\gamma_2^2 + 2\gamma_2\xi_2 \cos(\alpha-\alpha_z) + \xi_2^2]^{\frac{1}{2}} \\ \vdots \\ [\gamma_n^2 + 2\gamma_n\xi_n \cos(\alpha-\alpha_z) + \xi_n^2]^{\frac{1}{2}} \end{pmatrix} \tag{C.4}$$

Appendix D

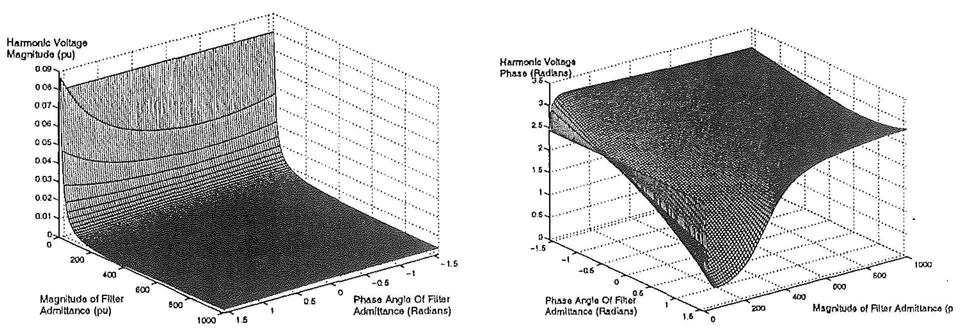
PASSIVE FILTER EXAMPLE

This appendix contians the plots detailing the variation in the magnitude and phase angle of the harmonic voltage at each busbar, due to variation in the filter installed at busbar one.



(a) Voltage magnitude V_{h1} (b) Voltage phase angle θ_1

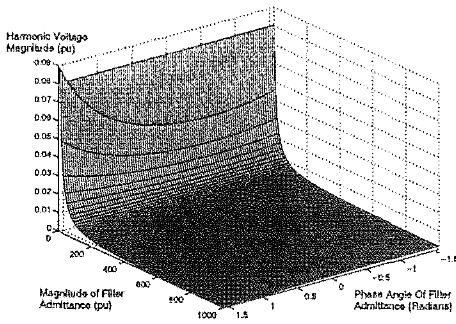
Figure D.1 Harmonic voltage variation at busbar one, due to variation of passive filter admittance



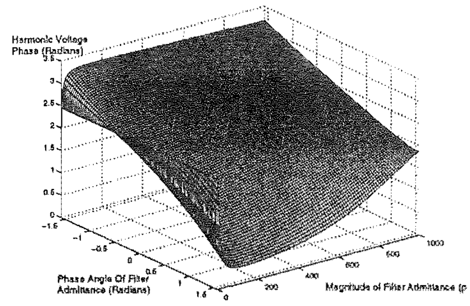
(a) Voltage magnitude V_{h2} (b) Voltage phase angle θ_2

Figure D.2 Harmonic voltage variation at busbar two, due to variation of passive filter admittance

Figures D.10-D.12 shows the voltage at each busbar as the magnitude of the filter admittance varies. In developing some the results of Chapter 6, assumptions were made as to the behaviour of the harmonic voltage at each busbar. Specifically it was stated, that at each busbar if the harmonic voltage is of a significant magnitude, the phase angle will be a consistent value. The following diagrams show this to be approxiamately true in that the phase at each busbar is

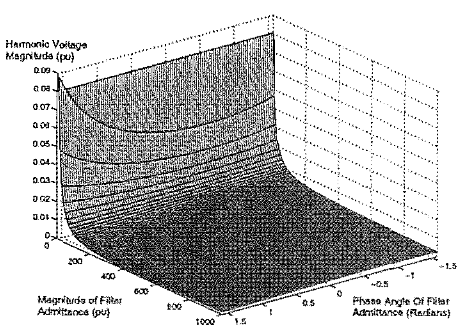


(a) Voltage magnitude V_{h3}

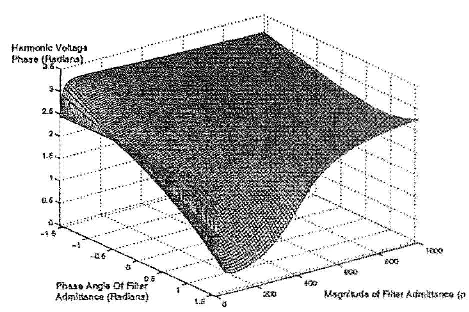


(b) Voltage phase angle θ_3

Figure D.3 Harmonic voltage variation at busbar three, due to variation of passive filter admittance

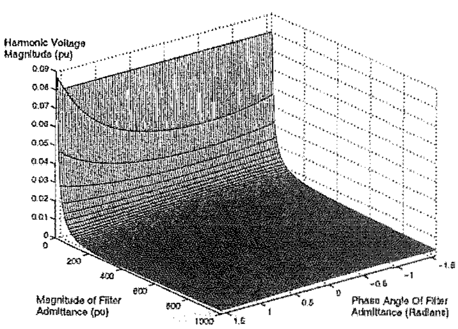


(a) Voltage magnitude V_{h4}

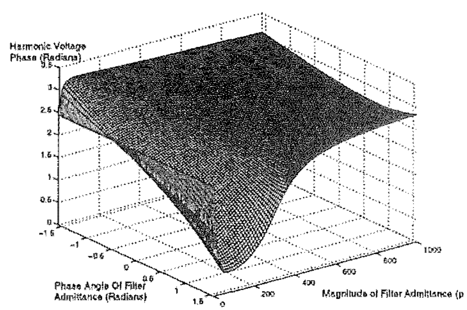


(b) Voltage phase angle θ_4

Figure D.4 Harmonic voltage variation at busbar four, due to variation of passive filter admittance

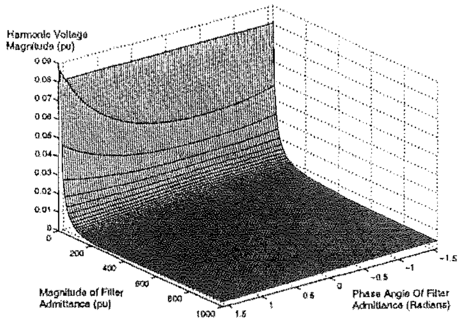


(a) Voltage magnitude V_{h5}

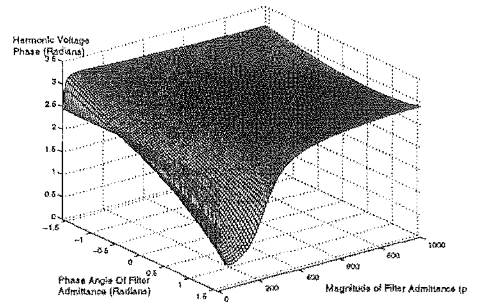


(b) Voltage phase angle θ_5

Figure D.5 Harmonic voltage variation at busbar five, due to variation of passive filter admittance

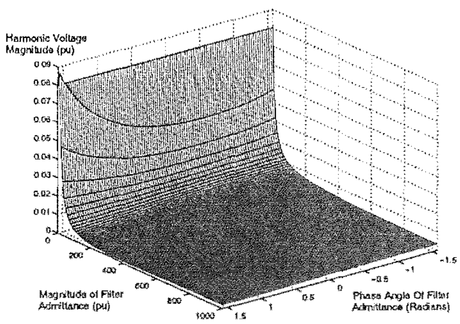


(a) Voltage magnitude V_{h6}

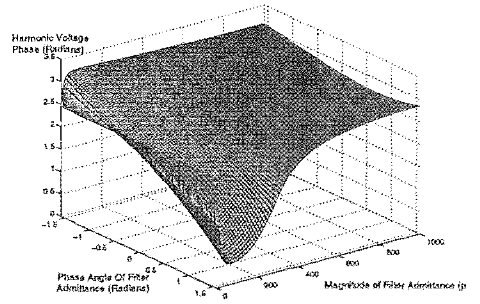


(b) Voltage phase angle θ_6

Figure D.6 Harmonic voltage variation at busbar six, due to variation of passive filter admittance

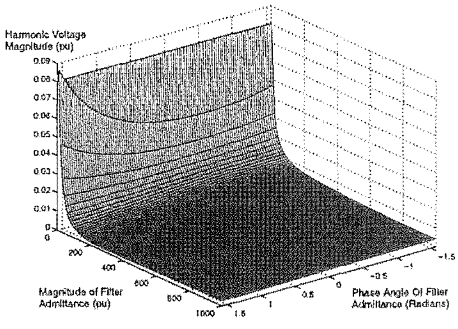


(a) Voltage magnitude V_{h7}

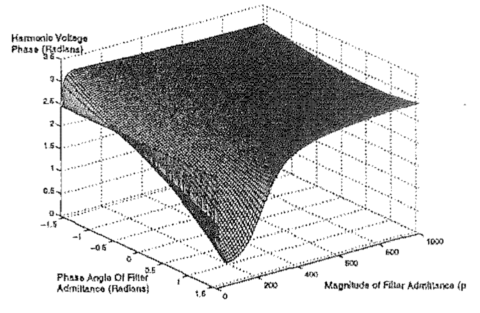


(b) Voltage phase angle θ_7

Figure D.7 Harmonic voltage variation at busbar seven, due to variation of passive filter admittance

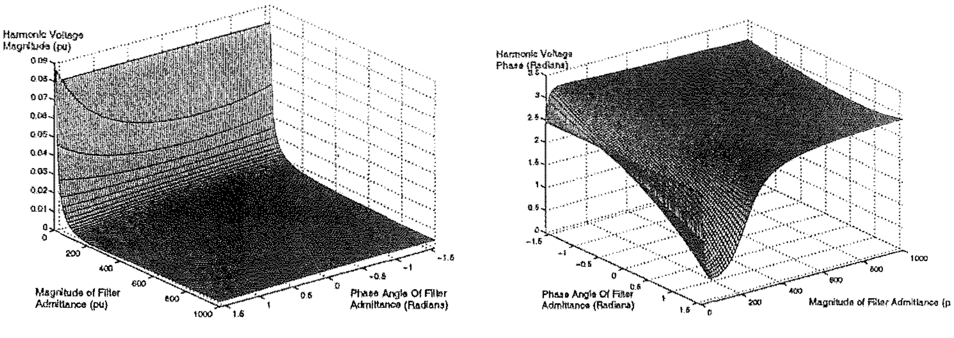


(a) Voltage magnitude V_{h8}



(b) Voltage phase angle θ_8

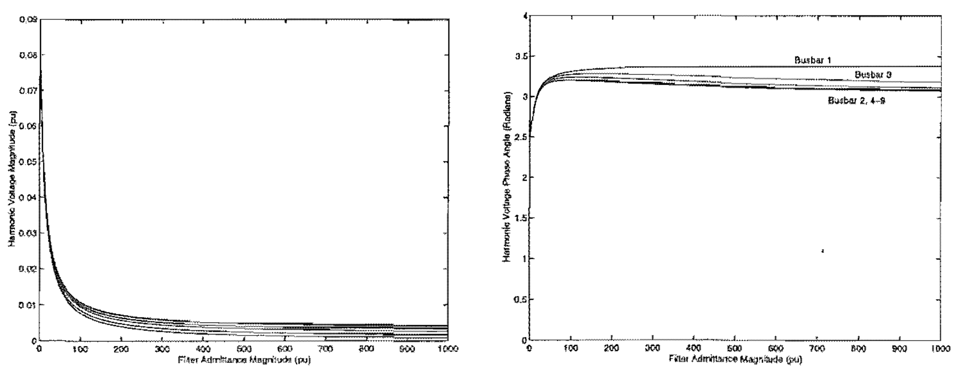
Figure D.8 Harmonic voltage variation at busbar eight, due to variation of passive filter admittance



(a) Voltage magnitude V_{h9} (b) Voltage phase angle θ_9

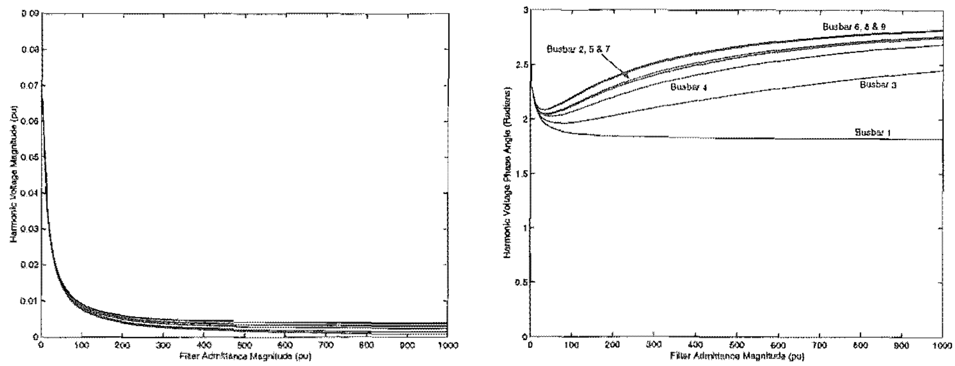
Figure D.9 Harmonic voltage variation at busbar nine, due to variation of passive filter admittance

equal until the filter becomes large enough that the voltage distortion drops towards zero. At this point the harmonic phase at different busbars does diverge.



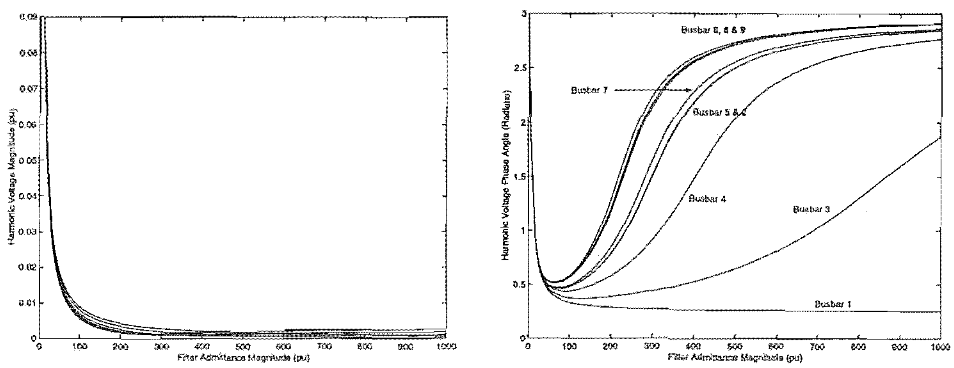
(a) Voltage magnitude $\mathbf{V_h}$ (b) Voltage phase angle $\boldsymbol{\theta}$

Figure D.10 Harmonic voltage variation at each busbar, due to variation of filter admittance magnitude, where the filter is inductive



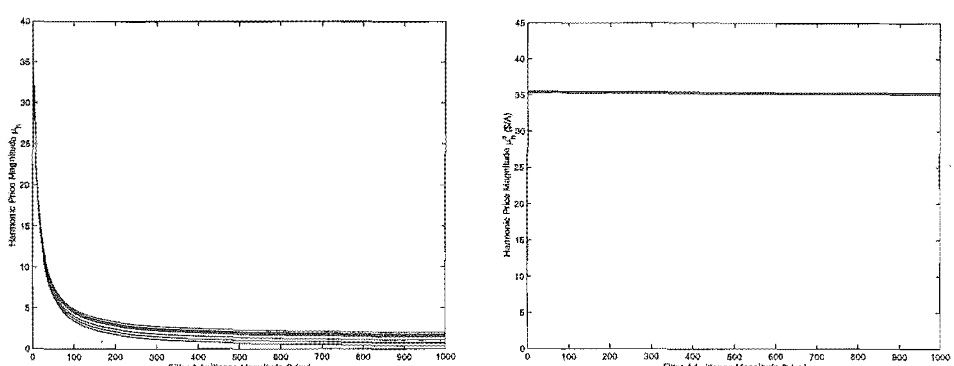
(a) Voltage magnitude $\mathbf{V}_\mathbf{h}$ (b) Voltage phase angle $\boldsymbol{\theta}$

Figure D.11 Harmonic voltage variation at each busbar, due to variation of filter admittance magnitude, where the filter is resistive



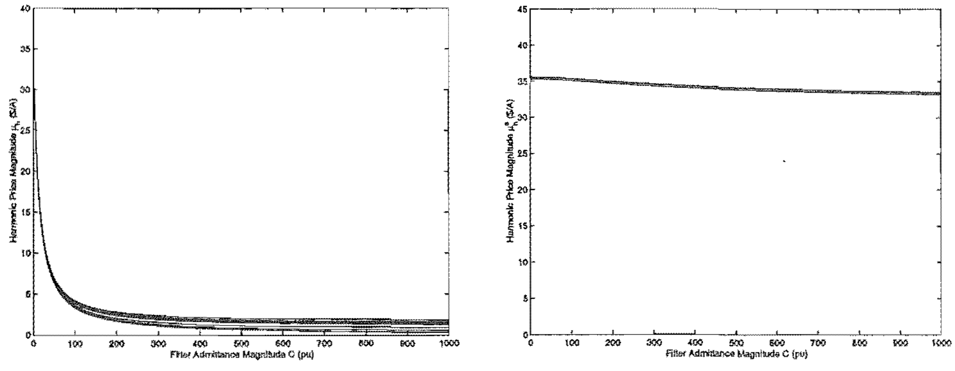
(a) Voltage magnitude $\mathbf{V}_\mathbf{h}$ (b) Voltage phase angle $\boldsymbol{\theta}$

Figure D.12 Harmonic voltage variation at each busbar, due to variation of filter admittance magnitude, where the filter is capacitive



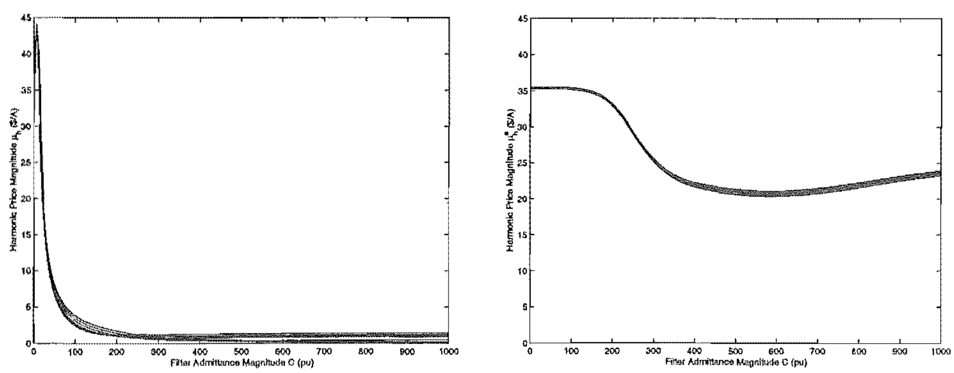
(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.13 Variation in the magnitude of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an inductive filter at busbar one is varied



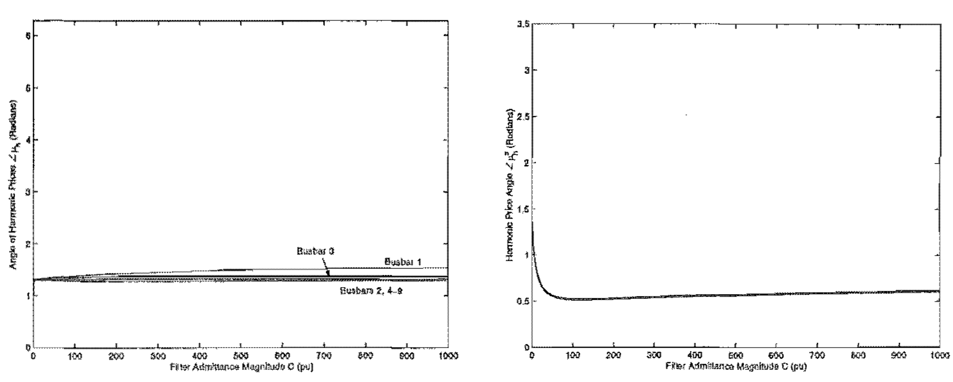
(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.14 Variation in the magnitude of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an resistive filter at busbar one is varied



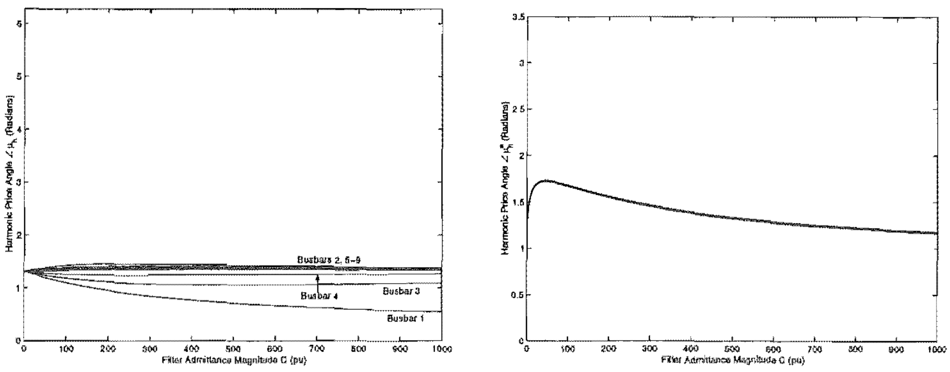
(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.15 Variation in the magnitude of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an capacitive filter at busbar one is varied



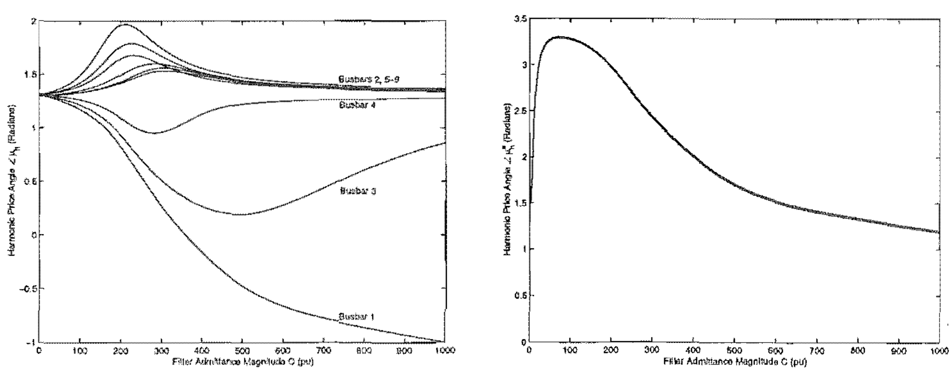
(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.16 Variation in the angle of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an inductive filter at busbar one is varied



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.17 Variation in the angle of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an resistive filter at busbar one is varied



(a) Harmonic Prices $\widetilde{\mu}_h$ (b) Harmonic Prices $\widetilde{\mu}_h^s$

Figure D.18 Variation in the angle of the two different harmonic prices $\widetilde{\mu}_h$ and $\widetilde{\mu}_h^s$, as the magnitude of an capacitive filter at busbar one is varied

Appendix E

POLAR TO CONJUGATE TRANSFORM

The nonlinear load dynamics are described using complex conjugate tensors. The harmonic source's sensitivity to the busbar voltage is given by the matrix Y_C.

$$Y_C = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix}$$

When investigating the affect on the harmonic prices, of variation in the sensitivity of injections to the voltage, changes in the matrices Y_1 and Y_2, are not very intuitive. But instead if the nonlinear load is described in polar form:

$$I_h = f_1(V_h, \theta)$$
$$\alpha = f_2(V_h, \theta)$$

So that

$$\begin{bmatrix} \Delta I_h \\ \Delta \alpha \end{bmatrix} = \begin{bmatrix} \frac{\partial I_h}{\partial V_h} & \frac{\partial I_h}{\partial \theta} \\ \frac{\partial \alpha}{\partial V_h} & \frac{\partial \alpha}{\partial \theta} \end{bmatrix} \begin{bmatrix} \Delta V_h \\ \Delta \theta \end{bmatrix} \tag{E.1}$$

Variation in the sensitivities of I_h and α, to V_h and θ, are intuitively easy to understand. Hence when investigating the behaviour of the harmonic prices to changes in the dynamic qualities of the nonlinear loads, it is preferable to work in the polar domain. As nodal circuit analysis is only linear in the conjugate domain, a transform what is required that can convert variation of polar parameters (intuitively understood), to variation, in the complex conjugate parameters.

$$\widetilde{V} = V e^{j\theta}$$
$$\widetilde{V}^* = V e^{-j\theta}$$

$$\begin{aligned} \therefore \begin{bmatrix} \Delta \widetilde{V} \\ \Delta \widetilde{V}^* \end{bmatrix} &= \begin{bmatrix} e^{j\theta} & jVe^{j\theta} \\ e^{-j\theta} & -jVe^{-j\theta} \end{bmatrix} \begin{bmatrix} \Delta V_h \\ \Delta \theta \end{bmatrix} \\ &= [Trans_\theta] \begin{bmatrix} \Delta V_h \\ \Delta \theta \end{bmatrix} \end{aligned} \tag{E.2}$$

$$[Trans_\theta]^{-1} = \begin{bmatrix} \frac{1}{2}e^{-j\theta} & \frac{1}{2}e^{j\theta} \\ -j\frac{1}{2V}e^{-j\theta} & j\frac{1}{2V}e^{j\theta} \end{bmatrix} \tag{E.3}$$

Equivalent matrices can be developed, which link variation in the complex conjugate current terms to variation in the current polar parameters ($[Trans_\alpha]$).

$$[Trans_\alpha]\begin{bmatrix} \Delta I_h \\ \Delta\alpha \end{bmatrix} = [Trans_\alpha]\begin{bmatrix} \frac{\partial I_h}{\partial V_h} & \frac{\partial I_h}{\partial \theta} \\ \frac{\partial \alpha}{\partial V_h} & \frac{\partial \alpha}{\partial \theta} \end{bmatrix}\begin{bmatrix} \Delta V_h \\ \Delta\theta \end{bmatrix} \tag{E.4}$$

$$[Trans_\alpha]\begin{bmatrix} \Delta I_h \\ \Delta\alpha \end{bmatrix} = [Trans_\alpha]\begin{bmatrix} \frac{\partial I_h}{\partial V_h} & \frac{\partial I_h}{\partial \theta} \\ \frac{\partial \alpha}{\partial V_h} & \frac{\partial \alpha}{\partial \theta} \end{bmatrix}[Trans_\theta]^{-1}\begin{bmatrix} \Delta\widetilde{V} \\ \Delta\widetilde{V}^* \end{bmatrix} \tag{E.5}$$

$$\begin{bmatrix} \Delta\widetilde{I}_h \\ \Delta\widetilde{I}_h^* \end{bmatrix} = \begin{bmatrix} e^{j\alpha} & jI_h e^{j\alpha} \\ e^{-j\alpha} & -jI_h e^{-j\alpha} \end{bmatrix}\begin{bmatrix} \frac{\partial I_h}{\partial V_h} & \frac{\partial I_h}{\partial \theta} \\ \frac{\partial \alpha}{\partial V_h} & \frac{\partial \alpha}{\partial \theta} \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{-j\theta} & \frac{1}{2}e^{j\theta} \\ -j\frac{1}{2V}e^{-j\theta} & j\frac{1}{2V}e^{j\theta} \end{bmatrix}\begin{bmatrix} \Delta\widetilde{V} \\ \Delta\widetilde{V}^* \end{bmatrix} \tag{E.6}$$

$$\Rightarrow \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_1^* \end{bmatrix} = \begin{bmatrix} e^{j\alpha} & jI_h e^{j\alpha} \\ e^{-j\alpha} & -jI_h e^{-j\alpha} \end{bmatrix}\begin{bmatrix} \frac{\partial I_h}{\partial V_h} & \frac{\partial I_h}{\partial \theta} \\ \frac{\partial \alpha}{\partial V_h} & \frac{\partial \alpha}{\partial \theta} \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{-j\theta} & \frac{1}{2}e^{j\theta} \\ -j\frac{1}{2V}e^{-j\theta} & j\frac{1}{2V}e^{j\theta} \end{bmatrix} \tag{E.7}$$

Equation E.7, is the transformation that converts variation in the polar parameters, to variation in the complex conjugate parameters.

Appendix F

PASSIVITY CRITERION

Passivity requires the admittance looking into any busbar has a positive real component.

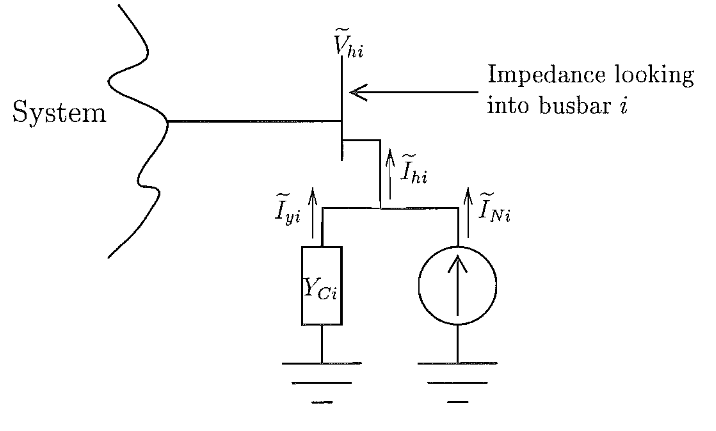


Figure F.1 Impedance looking into given busbar includes the local load and the rest of the system

In choosing polar parameters for the nonlinear loads it is possible to choose values, which fail to meet this criteria. As true power electronic loads are passive, one needs to be aware as to when the nonlinear load's parameters are specified so that the device ceases to behave realistically. In the example demonstrating the behaviour of marginal prices in response to changes in the nonlinear loads parameters, one must expect strange behaviour (resonances) should the system created cease to be passive and the system unstable. For instance if the parameters of the nonlinear load are created so that any harmonic injection is amplified to an infinitely large harmonic voltage, there will be a corresponding explosion in the prices and payments due from all distorting loads. Obviously infinitely large prices are ridiculous under normal circumstances, but in response to a ridiculous situation, they are what one would expect.

Requiring the real part of the impedance to be positive equates to requiring the system and load to absorb harmonic energy from any Norton injections. Passivity also needs to be defined for a tensor environment. The tensor representation of the system is based on the assumption the variables can be described via:

$$\Delta\widetilde{I}_h = \widetilde{y}_1\Delta\widetilde{V}_h + \widetilde{y}_2\Delta\widetilde{V}_h^* \tag{F.1}$$

So that if $\widetilde{y}_1$ and $\widetilde{y}_2$ are admittances:

$$\begin{aligned} \widetilde{I}_h &= \widetilde{y}_1\widetilde{V}_h + \widetilde{y}_2\widetilde{V}^* \\ \Rightarrow \frac{\widetilde{I}_h}{\widetilde{V}_h} &= \widetilde{y}_1 + \widetilde{y}_2\frac{\widetilde{V}^*}{\widetilde{V}_h} \end{aligned} \tag{F.2}$$

$$\therefore \text{ Tensor Admittance} = \widetilde{y}_1 + y_2 e^{j(\angle\widetilde{y}_2 - 2\theta)} \tag{F.3}$$

For if the nonlinear is to be passive under all conditions, the impedance tenor must be such that:

$$Re\{\widetilde{y}_1\} < y_2 \tag{F.4}$$

This impedance tensor can be shown graphical on the so called lollipop diagram (Figure F.2) [Smith1996].

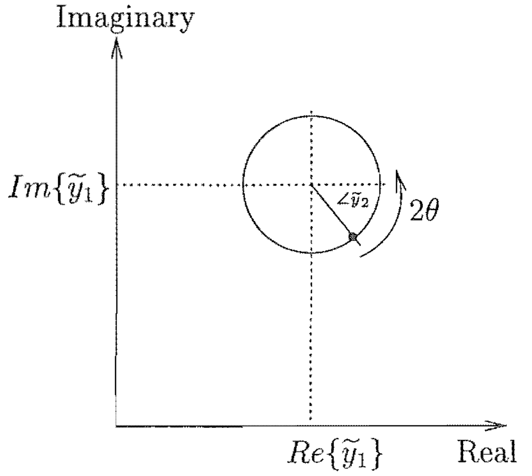


Figure F.2 Complex impedance locus for an impedance tensor

To find the tensor impedance looking into each busbar in the network, one can manipulate the full admittance matrix of the system. If the system nodal equation is rewritten:

$$\begin{bmatrix} \mathbf{I_i} \\ \mathbf{I_{not\,i}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a \end{bmatrix} & \begin{bmatrix} b \end{bmatrix} \\ \begin{bmatrix} c \end{bmatrix} & \begin{bmatrix} d \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{V_i} \\ \mathbf{V_{not\,i}} \end{bmatrix} \tag{F.5}$$

$$\text{Where} \quad \mathbf{I_i} = \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{hi}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{hi}} \end{bmatrix} \tag{F.6}$$

$$\mathbf{I_{not\,i}} = \begin{bmatrix} \widetilde{\mathbf{I}}_{\mathbf{h1}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{h1}} \\ \widetilde{\mathbf{I}}_{\mathbf{h2}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{h2}} \\ \vdots \\ \widetilde{\mathbf{I}}_{\mathbf{hn}} \\ \widetilde{\mathbf{I}}^*_{\mathbf{hn}} \end{bmatrix} \quad \text{All busbars execpt } i \tag{F.7}$$

$$\mathbf{V_i} = \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{hi}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{hi}} \end{bmatrix} \tag{F.8}$$

$$\mathbf{V_{not\,i}} = \begin{bmatrix} \widetilde{\mathbf{V}}_{\mathbf{h1}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h1}} \\ \widetilde{\mathbf{V}}_{\mathbf{h2}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{h2}} \\ \vdots \\ \widetilde{\mathbf{V}}_{\mathbf{hn}} \\ \widetilde{\mathbf{V}}^*_{\mathbf{hn}} \end{bmatrix} \qquad \text{All busbars execpt } i \tag{F.9}$$

Setting $\mathbf{I_{not\,i}} = \mathbf{0}$, one can derive the admittance looking into busbar i.

$$\frac{\partial \mathbf{I_i}}{\partial \mathbf{V_i}} = [a] - [b][d]^{-1}[c] \tag{F.10}$$

Appendix G

DYNAMIC NONLINEAR LOAD EXAMPLE

In this appendix the test system of Appendix A is used. The behaviour of the Norton ($\widetilde{\mu}_N$), and harmonic injection prices ($\widetilde{\mu}_T$) are investigated, as the parameters of the impedance tensor for the load at busbar four are varied. Also the variation in the harmonic payments made and received by the loads is detailed. This example is identical to the example in Chapter 7 in the polar parameters of the load at busbar are varied. Specifically the sensitivity of injected current magnitude, to changes in harmonic voltage magnitude ($\frac{\partial I_{h4}}{\partial V_{h4}}$) is varied. All the other parameters are set equal to zero.

$$\begin{bmatrix} \Delta I_{h4} \\ \Delta \alpha_4 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{h4} \\ \Delta \theta_4 \end{bmatrix} \quad \text{Where } k = 0 \rightarrow 20 \tag{G.1}$$

This variation in the load's polar parameters, is easily transformed to variation in its complex conjugate tensor parameters, using the transform detailed in Appendix E. All the other loads are modelled as static harmonic current sources, as before.

G.1 CONSTANT HARMONIC INJECTION

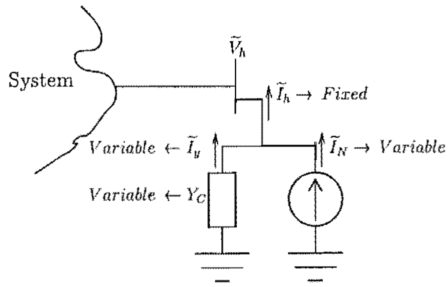
In this example the injected harmonic currents detailed in Appendix A, are specified as the harmonic injections for each load. Except for the nonlinear load at busbar four, these harmonic injections are also the Norton injections for each load. The nonlinear load at busbar four is modelled as a Norton equivalent, where the impedance parameters are varied (as described above). As a result the harmonic injection for each the load is constant, but the Norton injection from the load at busbar four, will vary with the change in that load's tensor impedance (Figure G.1).

As the total harmonic injections into the network are constant, the harmonic voltage throughout the network is also constant (Fig. G.2). As such the harmonic compensation paid to each load ($k_i V_{hi}$) and in total ($\mathbf{K}\mathbf{V}_\mathrm{h}$), are also constant (Fig. G.3).

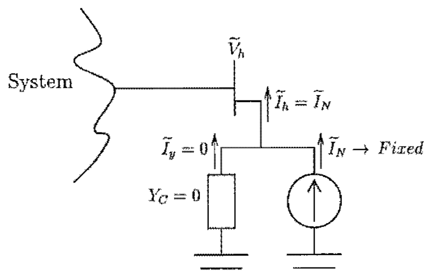
For all the nonlinear loads, apart from that at busbar four, the harmonic injections are equal to the Norton injections and are constant. For the load at busbar four, the Norton injections of the load vary as its polar parameters are varied. This variation in the Norton injections are shown in Fig. G.4.

Figure G.5, shows the price for Norton injections. Again there is a resonance at the point where the network switches from being passive to active. Unlike the example in Chapter 7, this resonance in the Norton prices is not associated with a resonance in the harmonic voltage.

Figures G.6 and G.7, show the amount each load is charged for their Norton injections, when the prices $\widetilde{\mu}_N$ are used. Where $\frac{\partial I_{h4}}{\partial V_{h4}} \approx 4$, there is no Norton injection into the network by the load at busbar four. At the same point the Norton payments made by the load are zero.

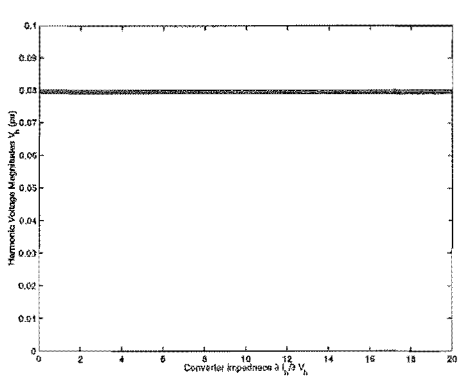


(a) Busbar four

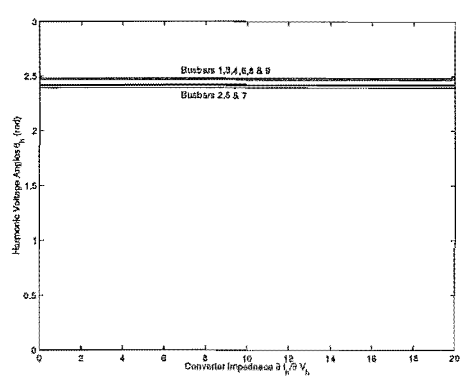


(b) All other busbars

Figure G.1 Norton equivalent models of the nonlinear loads used in the example

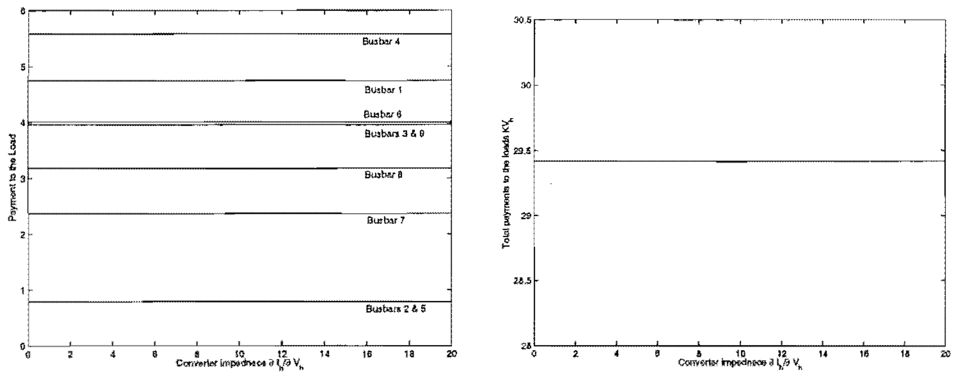


(a) Voltage Magnitude V_{hi}



(b) Voltage Angle θ_i

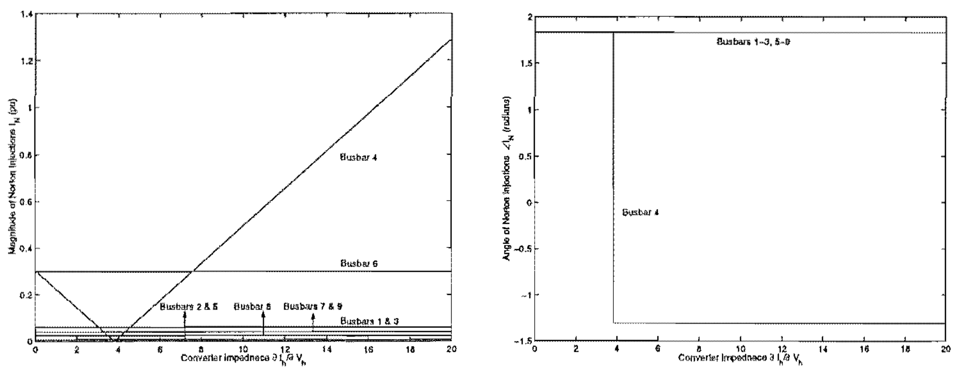
Figure G.2 Magnitude and angle of harmonic voltage at each busbar as parameters of busbar four are varied



(a) Individual Payments $k_i V_{hi}$

(b) Total Payments $\mathbf{KV_h}$

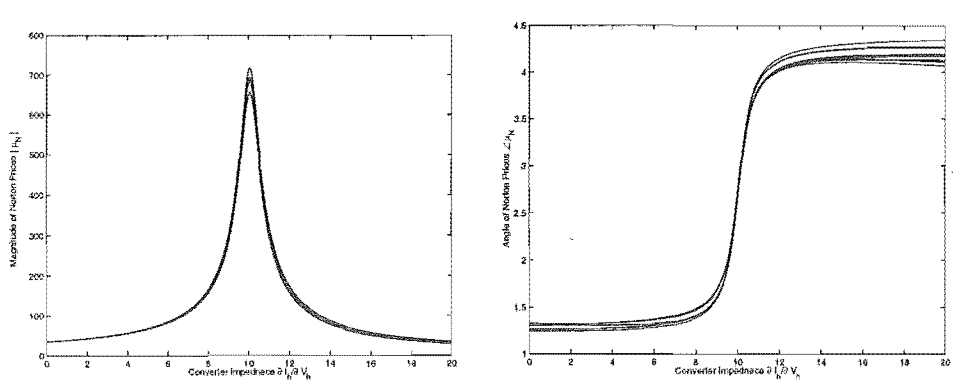
Figure G.3 Harmonic distortion compensation paid to each load, and in total as the parameters of busbar four are varied



(a) Current Magnitude $\|\widetilde{\mathbf{I}}_\mathbf{N}\|$

(b) Current Angle $\angle\widetilde{\mathbf{I}}_\mathbf{N}$

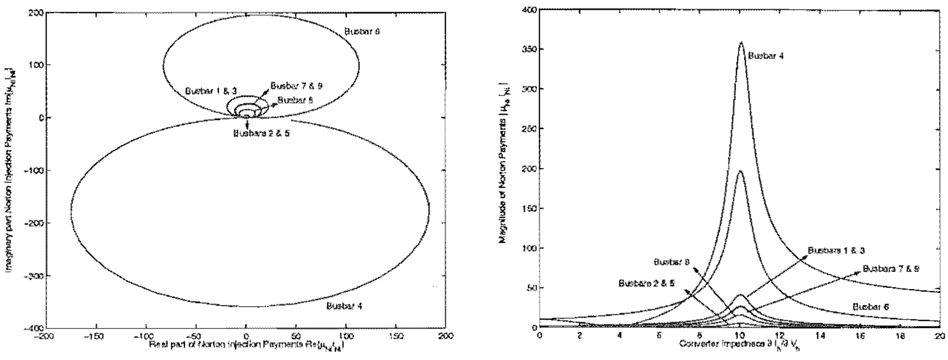
Figure G.4 Norton current injections by each load into the network, as the parameters of busbar four are varied



(a) Price Magnitude $\|\widetilde{\mu}_N\|$

(b) Price Angle $\angle\widetilde{\mu}_N$

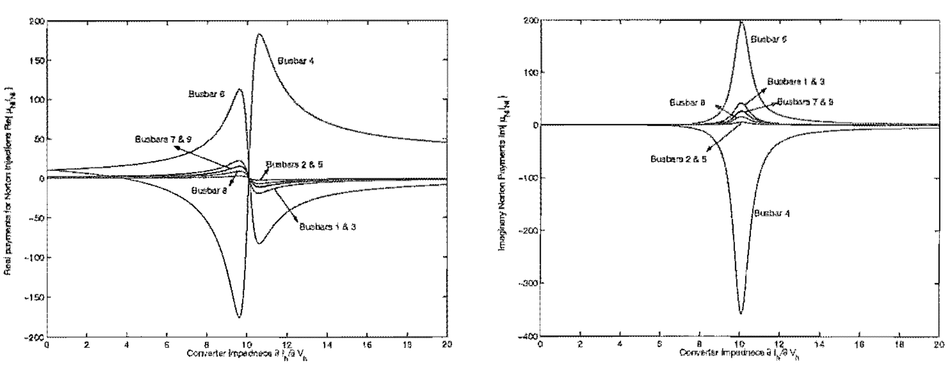
Figure G.5 Variation in the price for Norton injections into the network, as the parameters of busbar four are varied



(a) Complex Norton Payments $\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}$

(b) Magnitude of Norton Payments $||\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}||$

Figure G.6 Norton injection payments made by each load, as the parameters of busbar four are varied

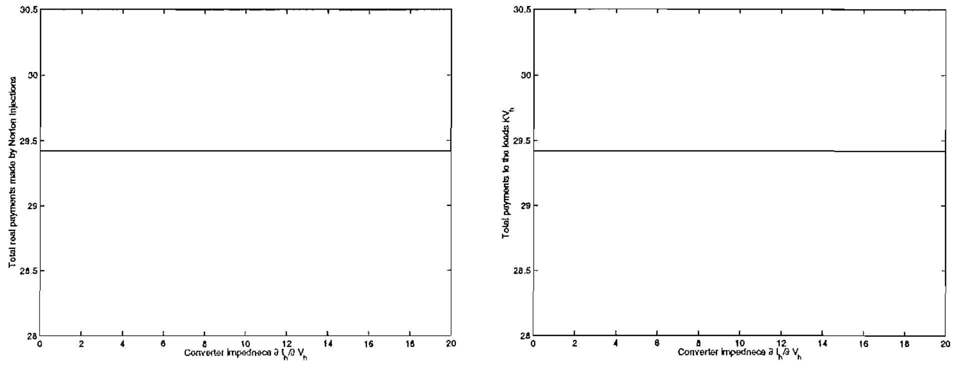


(a) Real Norton Payments $Re\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}$

(b) Imaginary Norton Payments $Im\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}$

Figure G.7 Real and imaginary parts of the complex Norton charges faced by each load, as the parameters of busbar four are varied

Figure G.8, shows the summation of the real component of the Norton charges ($\sum Re\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}$), is equal to the value the aggregate system places on the voltage distortion. Evidence that the real component of the charges, indicates the value the network places on the injections.



(a) Individual Norton Payments $\sum Re\{\widetilde{\mu}_{Ni}\widetilde{I}_{Ni}\}$

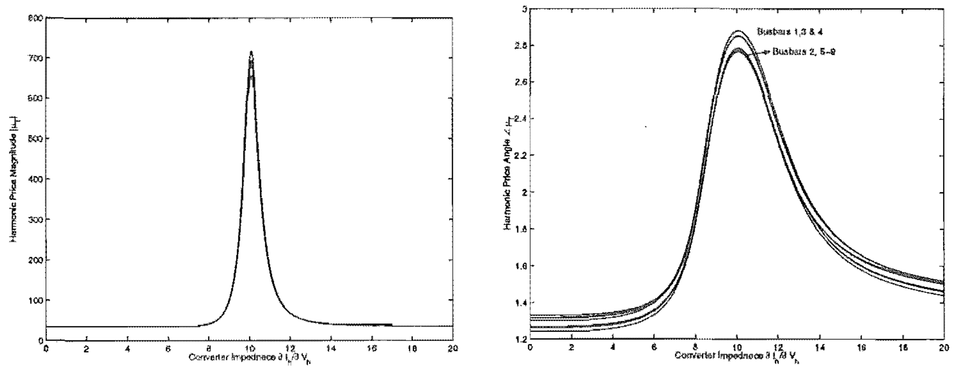
(b) Total Payments $\mathbf{KV_h}$

Figure G.8 Total real Norton payments made by each load, and the total harmonic compensation payments due, as parameters of busbar four are varied

Figure G.9, details the prices for harmonic injections ($\widetilde{\mu}_T$), as the parameters of the load at busbar four are varied. These prices have a very similar magnitude to those for the Norton injections into the Network. This is because the total Norton injections and the total Harmonic injections into the network are very similar. This was not the case in the Chapter 7 example, where the total harmonic injections were many times larger than the total Norton injections, and as such the prices for the harmonic injections were much smaller than the prices for the Norton injections. In Chapter 7, it was shown the prices for the harmonic injections ($\widetilde{\mu}_T$) are equivalent to the to the price for the Norton injections ($\widetilde{\mu}_N$), if the passive filter and dynamic components of the load are excluded. But this interpretation only holds if the nonlinear load current is a linear function of the voltage distortion present (does not require tensor representation). In this example variation in filter injection is a tensor function of the voltage, and as such the price for the harmonic injections is a function of the load variation. If the harmonic injection was a straight linear function of the harmonic voltage (i.e. $\widetilde{\mathbf{I}}_{\mathrm{h}} = \widetilde{\mathbf{I}}_{\mathrm{N}} + [Y_1]\widetilde{\mathbf{V}}_{\mathrm{h}}$), these prices would have been independent of the parameter variation at busbar four.

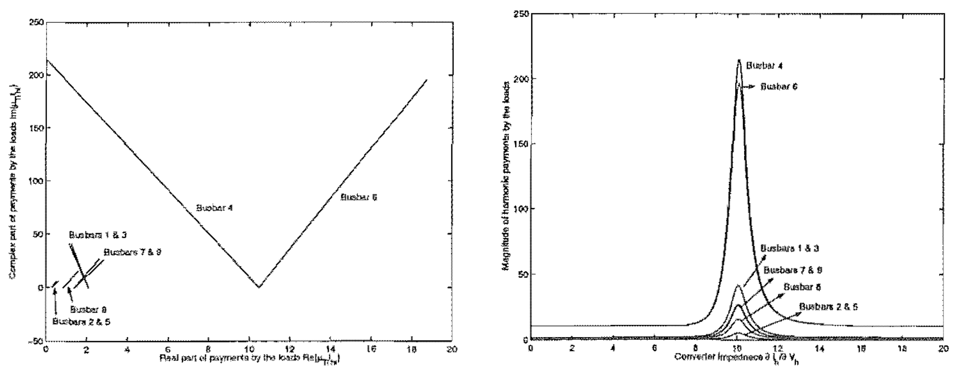
Figures G.10 and G.11, show the variation in the harmonic charges faced by each load. Of particular interest is that the real component of the charges (which represents the true value of the harmonic injections), is negative for some of the loads, despite that the loads' injections are constrained to a common phase angle. The harmonic injection of each load have similar consequences for the network, but yet some loads pay for their injections and others are rewarded. This demonstrates how the true value of the injections into the network is represented by the price for the Norton injections. As such the prices for harmonic injections while convenient, can provide inefficient signals.

Figure G.12 demonstrates the summation of the real component of the harmonic charges is equal to the total amount due to loads in the way of voltage distortion compensation. This is the main advantage of using the adjusted harmonic charges ($\widetilde{\mu}_T$), the books will balance if loads are charged based on their total harmonic injections.



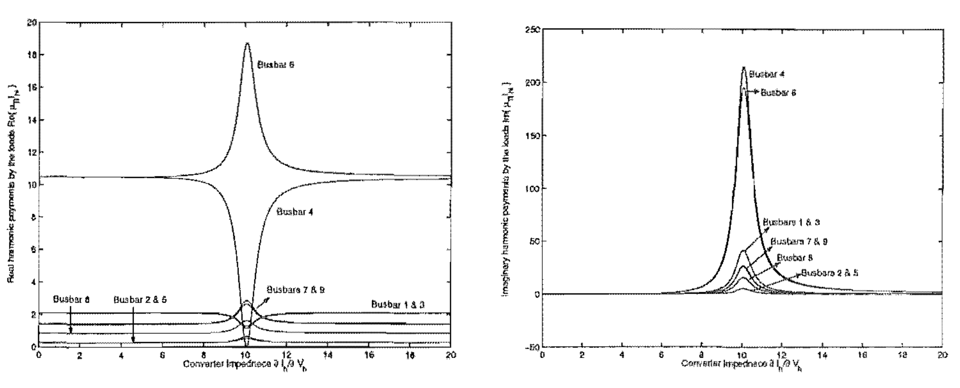
(a) Price Magnitude $\|\widetilde{\mu}_T\|$ (b) Price Angle $\angle\widetilde{\mu}_T$

Figure G.9 Variation in the price for harmonic injections into the network, as the parameters of busbar four are varied



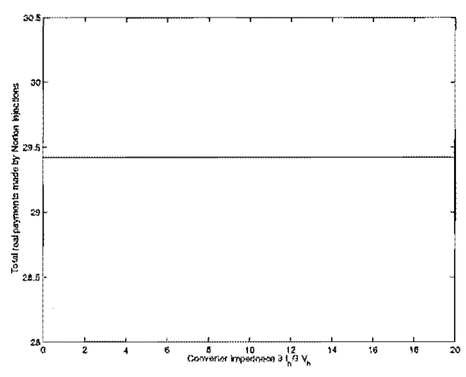
(a) Complex Harmonic Payments $\widetilde{\mu}_{Ti}\widetilde{I}_{hi}$ (b) Magnitude of Harmonic Payments $\|\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\|$

Figure G.10 Harmonic payments made by each load, as the parameters of busbar four are varied

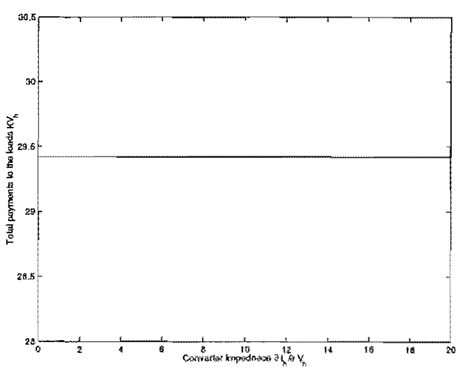


(a) Real Harmonic Payments $Re\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}$ (b) Imaginary Harmonic Payments $Im\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}$

Figure G.11 Real and imaginary parts of the complex harmonic charges faced by each load, as the parameters of busbar four are varied



(a) Individual Harmonic Payments $\sum Re\{\widetilde{\mu}_{Ti}\widetilde{I}_{hi}\}$



(b) Total Payments $\mathbf{KV_h}$

Figure G.12 Total real harmonic payments made by each load, and the total harmonic compensation payments due, as parameters of busbar four are varied

Appendix H

PUBLISHED PAPERS

The following papers have been published in conjunction with this work

1. P. Talacek, N. Watson, 'Marginal pricing of harmonic injections', *IEEE Transactions on Power Systems*, vol. 17, no. 1, pp. 50-56, February 2002
2. P. Talacek, N. Watson, 'Valuation of harmonic injections', *Electrical Power and Energy Systems (submitted for publication)*, 2001
3. P. Talacek, N. Watson, 'Marginal pricing of harmonic injections: An analsis of the resulting payments', *IEEE Transactions on Power Systems (accepted for publication)*, July 2002
4. P. Talacek, N. Watson, 'Filter Resource Allocation Using Marginal Pricing of Harmonic Injections', *IEEE Transactions on Power Systems (accepted for publication subject to changes)*, January 2002

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